KIMBER'S CONSTRUCTIONS

QUESTIONS TO

B.A. AND B. S.C.

AND

SOLUTIONS

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GRADUATION MATHEMATICS

COMPRISING

FORTY-ONE YEARS' QUESTIONS IN ARITHMETIC AND ALGEBRA

SET AT THE UNIVERSITY OF LONDON

FOR THE DEGREES OF

B.A. AND B.Sc.

COMPLETE FROM 1839 TO 1879

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BY

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PASS EXAMINATION PAPERS,

FOR

THE DEGREE OF BACHELOR OF ARTS.

1839. *Monday, May 27th.*—Examiner,—Mr. Jerrard.

1. In what does the peculiar excellence of the present system of numerical notation consist?

2. Find the value of $\frac{355}{113}$ to six places of decimals, and of $\frac{0\cdot01331}{1\cdot3173}$ to four places.

3. Solve the following equations:

$$(1)\quad 2x - \frac{1}{2}(x+3) = 6.$$

$$(2)\quad (x+2)(x+3) = x(x+4).$$

$$(3)\quad x^2 - 4x + 3 = 0.$$

$$(4)\quad \sqrt{x^2 - 2x + 93} - \frac{x^2}{2} = 45 - x.$$

4. Prove that cos. $2\theta = 1 - 2(\sin.\ \theta)^2$.

5. In every plane triangle the sides are as the sines of the opposite angles.

6. Find the equation to the ellipse referred to rectangular axes, the origin being at the centre.

(*Wednesday, May 29th.*)

7. Find the area of a rectangular court, of which the diagonal is 100 yards, and the breadth 42 yards.

8. Solve the following equations:

$$(1)\quad \sqrt{x} + \sqrt{(10+x)} = \frac{20}{\sqrt{(10+x)}}.$$

$$(2)\quad \begin{cases} x^3 + y^3 = 1001, \\ x + y = 11. \end{cases}$$

9. Given two sides and an included angle of a triangle, express the third side in a form adapted to logarithmic computation, the tabular radius being 10^{10}.

10. Find the equation to an ellipse referred to the axis minor and the tangent at its extremity as axes.

1840. *Monday, May 24th.*—Examiner,—Mr. JERRARD.

1. What extension takes place in the meaning of the term Multiplication, when applied to fractions? Investigate the general rule for multiplying any number of fractions together, and find the product of the decimals ·0101 and ·0202.

2. Explain the ordinary method of extracting the square root of a number. What is the square root of 2085·7489?

3. Find the simple interest of £352 11*s.* 2*d.* for 21 years and 3 months, at $3\frac{1}{4}$ per cent. per annum.

4. Solve the following equations:

$$(1)\quad \frac{5x-7}{2}-\frac{2x+7}{3}=3x-14.$$

$$(2)\quad \frac{x-1}{x-2}=\frac{x-3}{x-4}+1.$$

$$(3)\quad 2x+3y=37,\quad \frac{1}{x}+\frac{1}{y}=\frac{14}{45}.$$

There are three pipes, one of which fills a reservoir in 4 hours, another in $3\frac{1}{2}$ hours, and the third in $2\frac{1}{4}$ hours; what time will they take to fill it when they are all flowing together?

5. Sum the series

$$(1)\quad \frac{1}{3}+\frac{2}{3}+1+\ldots \text{ to 50 terms.}$$

$$(2)\quad \frac{2}{5}+\frac{3}{5^2}+\frac{2}{5^3}+\frac{3}{5^4}+\ldots \textit{ ad infinitum.}$$

6. Find the number of combinations that may be formed with m quantities, taken n at a time.

7. Show that $\log.(a\,b\,c\,d\ldots)=\log. a+\log. b+\log. c+\log. d+\ldots$

What are the advantages of Briggs's system of logarithms?

8. Define the sine and cosine of an angle. Also prove that

$$(1)\quad (\sin. A)^2+(\cos. A)^2=1.$$

$$(2)\quad \cos. A=\frac{1}{\sqrt{1+(\tan. A)^2}}.$$

9. Investigate the general formula

$$\text{Sin.}\,(A-B)=\sin. A\,.\cos. B-\cos. A\,\sin. B,$$

and show how it is verified when

$$\text{I. } B=0.\quad \text{II. } A=B.\quad \text{III. } A=\frac{\pi}{2}.$$

10. Express the area of a triangle in terms of its sides.

(*May 27th.*—Mr. MURPHY.)

11. Find the equation to a circle, when the origin is a point in

the circumference, and the axis of x a diameter passing through that point. Considering the ellipse as the Orthographic Projection of a circle, deduce its equation from that of the circle.

1841. *Monday, May* 31*st*.—Examiner,—Mr. JERRARD.

1. State the nature of the questions to which the *Rule of Three* is applicable; and show that a rule similar in principle may be applied to questions involving more than three quantities.

How many men can complete a trench of 468 yards in 8 days, if 24 men can dig 81 yards in 6 days?

2. Find the value of $\dfrac{\sqrt{\cdot 012}}{\sqrt{2}+\sqrt{3}}$ to 4 places of decimals.

3. Given log. 3 = ·4771213, log. 7 = ·8450980, find the logarithm of 1323 and that of 1·323. 1323 is equal to $3^3 \times 7^2$. What is the criterion of the divisibility of a number by 3?

4. In what time will a sum of money double itself at $3\frac{1}{2}$ per cent. per annum compound interest?

$$\text{Log. } 2 = \cdot 3010300,$$
$$\text{log. } 1\cdot 035 = \cdot 0149403.$$

5. Determine by actual multiplication the expansion of $(1-2b)^5$. Divide $a - b^2$ by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$. Also find the series which results from the division of 1 by $1 + x$.

What is the most simple form to which $\dfrac{3x^2+6x+3}{2x^3+2}$ can be reduced?

6. Solve the equations:

(1) $\dfrac{x}{2}+\dfrac{x}{3}-\dfrac{x}{4}=7.$ (2) $\dfrac{1}{1+x}+\dfrac{1}{1-x}=\dfrac{8}{3}.$

(3) $\left.\begin{aligned}2x+5y&=89,\\ 7x-8y&=31.\end{aligned}\right\}$ (4) $\left.\begin{aligned}xy&=x-y,\\ x+y&=1.\end{aligned}\right\}$

(5) $x^2 - 3x - 130 = 0.$

(6) $7\sqrt{2x^2-10x+3} = 72 + 5x - x^2.$

7. How many variations can be made of the letters in the word *language?*

8. The sum of a decreasing arithmetic series is 140, the first term 10, and the common difference $\frac{1}{3}$; find the number of terms.

Find the sum of 15 terms of the series $1+\dfrac{1}{3}+\dfrac{1}{9}+\dfrac{1}{27}+\ldots$

9. Assuming the expression for sin. $(\alpha+\beta)$, and that for cos. $(\alpha+\beta)$, prove that

$$(1)\quad \tan.(\alpha+\beta) = \frac{\tan.\alpha+\tan.\beta}{1-\tan.\alpha\tan.\beta}.$$

and apply it to deduce the equation

$$(2)\quad \tan.(45^\circ + \beta) = 2\tan. 2\beta + \tan.(45^\circ - \beta),$$

for determining the tangents of angles greater than 45°.

10. Show that in any plane triangle $c^2 = a^2 + b^2 - 2ab\cos. C$, where a, b, c, are the sides, and C is the angle opposite to c. How may this expression be adapted to logarithmic calculation?

(*June 2nd.*—Rev. R. Murphy.)

11. Find the equation to an ellipse referred to its axes major and minor.

1842. *Monday, Oct. 3rd.*—Examiner,—Mr. Jerrard.

1. Explain fully what is meant by a fraction, and show that the value of a fraction is not altered by multiplying the numerator and denominator by the same number. What decimal of £1 is $\frac{3}{8}$ of a guinea?

2. A person travelled from Slough to London, a distance of $18\frac{1}{2}$ miles, in twenty-five minutes; at what rate is this per hour?

How much would 456 tons cost at £3 17*s.* $5\frac{3}{4}$*d.* per ton?

3. Find the continued product of

$$x - a,\ x + a,\ x - a\sqrt{-1},\ x + a\sqrt{-1},$$

and reduce $\dfrac{x^3 - 6x^2 - 37x + 210}{x^3 + 4x^2 - 47x - 210}$ to its most simple form. What is the value of this fraction, when $x = 7$?

4. Solve the equations,

(1) $\dfrac{x}{7} - \dfrac{x-5}{11} + 5 = x - \left(\dfrac{2x}{77} + 1\right)$.

(2) $\left.\begin{array}{r} 3x + 11y = 84 \\ 7x - 19y = 62 \end{array}\right\}$. (3) $(5x - 3)^2 - 7 = 44x + 5$.

(4) $\sqrt{x - 8} = 2 + \sqrt{\tfrac{1}{2}x}$. (5) $\left.\begin{array}{r} x + y = 18 \\ x^3 + y^3 = 4914 \end{array}\right\}$.

When will the hour and minute hands be at right angles to each other between twelve and one o'clock?

5. Show that $\log. \dfrac{a}{b} = \log. a - \log. b$, $\log. a^m = m \log. a$.

Given $\log. 2 = \cdot 3010300$, find the logarithm of 5, and that of 2000.

6. Define the sine and cosine of an angle, and show that

$$\text{Sin.}(A + B) = \sin. A \cos. B + \cos. A \sin. B,$$

where A and B are any two angles.

7. In any triangle of which A, B, are two angles, and a, b, the opposite sides, prove that $\dfrac{a}{b} = \dfrac{\sin. A}{\sin. B}$. What will be the ratio of a to b, if $A = 45^\circ$, $B = 30^\circ$? Given a, b, A, discuss the ambiguity which may arise in determining the triangle.

8. Express, in a form adapted to calculation by logarithms, the area of a triangle in terms of the sides. What is the area of the triangle of which the sides are 3, 5, and 7 feet in length?

(*Oct. 5th.*—Rev. R. MURPHY.)

9. Find generally the equation to the common parabola referred to rectangular co-ordinates, the origin being arbitrary.

1843. *Monday, Oct. 2nd.*—Examiner,—Mr. JERRARD.

1. Reduce $\frac{1}{17}$ to a decimal, and extract the square root of ·00007038409.

2. Explain the nature and use of logarithms, and point out the advantages of the common system over that of Napier.

3. Prove the truth of the rule for the signs in multiplication. Find the fifth power of $(2a - 3b)$, and multiply $x^{\frac{2}{3}} + x^{\frac{1}{3}} y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} - y^{\frac{1}{3}}$.

4. Investigate the rule for finding the greatest common measure of two algebraic quantities, and apply it to reduce

$$\frac{x - 17x^2 + 79x - 63}{x^3 - 13x^2 + 15x + 189}$$

to its lowest terms.

5. Find the expression for the sum of n terms of the series,

$$a + ar + ar^2 + \ldots,$$

and sum the series,

$$2 + 6 + 18 + \ldots \text{ to 12 terms,}$$

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots \textit{ ad infinitum,}$$

$$1 + 1\tfrac{1}{2} + 2 + \ldots \text{ to 29 terms.}$$

6. Solve the equations,

(1) $\frac{7x + 5}{4} - \frac{x - 8}{3} = \frac{11x}{5}$.

(2) $\left.\begin{aligned} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2 \end{aligned}\right\}$. (3) $\left.\begin{aligned} x + y &= 21 \\ xy &= 61 \end{aligned}\right\}$.

(4) $x^2 - 3x + 5 = 7\sqrt{x^2 - 5x + 6} + 2x + 17$.

Required three numbers in geometric progression whose sum is 91, and the sum of their squares 4459.

7. How many degrees, minutes, &c., in the centesimal division of the quadrant, correspond to 101° 2′ 34″ in the nonagesimal system?

8. Given the sines and cosines of two angles, find the sine and cosine of (1) the sum and (2) the difference of the angles.

(*Oct. 5th.*—Rev. Prof. HEAVISIDE.)

9. Find the general equation to a circle referred to rectangular co-ordinates. How does the equation become modified by taking the origin, (1) in the circumference, (2) at the centre of the circle?

1844. *Monday, Oct. 7th.*—Examiner,—Mr. JERRARD

1. Reduce $\frac{13}{625}$ and $\frac{5}{11\frac{1}{2}}$ to decimal fractions. Divide 53·796 by 7·82, and explain the process.

2. If 17 cwt. 3 qrs. 11 lbs. cost £256 15*s.* 7*d.*, what will 3 cwt. 1 qr. 21 lbs. cost?

3. Expand $(a + 2b - 3c)^3$. Divide $a^2 - b^2$ by $a + b$. Also find the quotient of $\frac{a^2+b^2}{a+b}$ to five terms. What is the meaning of such a result as $(a + b\sqrt{-1})(a - b\sqrt{-1}) = a^2 + b^2$?

4. Required the number of permutations of n things taken m and m together.

5. Show how to find the sum of any number of terms (1) of an arithmetic, (2) of a geometric series.

Examples. $1 + 5 + 9 + \ldots$ to 32 terms.

$\frac{1}{5} + \frac{1}{3} + \frac{5}{9} + \ldots$ to 17 terms.

6. Find the value of x in the equation,

$$\frac{5x+7}{9} = \frac{4x}{13} - \frac{x-10}{3} + 5.$$

Explain the method of solving the quadratic equation,

$$x^2 + A_1 x + A_2 = 0,$$

and show that, if its roots be denoted by x_1 and x_2, we shall have

$$x_1 + x_2 = -A_1, x_2 = A_2.$$

Take as an example, $x^2 + 6x - 55 = 0$.

Also find x and y from the equations,

$$x^{x+y} = y^{4a}, \quad y^{x+y} = x^a,$$

by means of logarithms.

7. Define the sine, cosine, secant, tangent, and versed sine of an angle, and prove that $\tan. (A + B) = \frac{\tan. A + \tan. B}{1 - \tan. A \tan. B}$. What does this expression become, when $A = 45°$, $B = 30°$?

8. Express the sine of an angle of a triangle in terms of the sides, and in a form adapted to logarithms. Give a general account of the method of observing angles; and show how to find the height and distance of an inaccessible object on a horizontal plane.

(*Oct. 9th.*—Rev. Prof. HEAVISIDE.)

9. Find the equation to the ellipse referred to rectangular co-ordinates, the origin being taken either at the centre, or at the extremity of its major axis.

1845. *Monday, Oct. 6th.*—Examiner,—Mr. Jerrard.

1. Express $\frac{3}{4}$ as a decimal. What is the form of fractions which are convertible into finite decimals? Show that the recurring periods for $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \ldots \frac{6}{7}$, will consist of the same digits.

2. Two persons have gained in trade £375; the one put in £500, and the other £850: what is each person's just share of the profit? What will the rates of a parish of which the rental is £2154 15*s.* 6*d.* amount to, at 7½*d.* per pound?

3. Explain the rule of signs in the multiplication of algebraic quantities. Expand $(x + a)(x + b)(x + c)$.

What will this expression become, when

$$(1) \quad a = b = c. \qquad (2) \quad b = -a, c = 0?$$

Also divide $x^3 - 4x^2 - 7x + 10$ by $x^2 + x - 2$.

4. Place three arithmetic means between 1 and 7.

Solve the equations:

$$(1) \quad \frac{1}{4} - (4x + 3) = \frac{x}{3} - 6. \qquad (2) \quad \left.\begin{array}{l} 5x + 6y = 89 \\ 12x - 7y = 21 \end{array}\right\}.$$

$$(3) \quad x^2 + 14x - 51 = 0;$$

and explain the process for the solution of quadratic equations.

5. Show that the number of permutations of n things, taken r and r together, is

$$n(n-1)(n-2)\ldots(n-r+1).$$

How many distinct permutations can be formed with the letters in the word *degree?* What is the general expression for the number of permutations, when there are different sets or classes of identical letters?

6. Define the term *logarithm.* What is meant by the *mantissa?* Given

$$\text{Log. } 2 = \cdot 3010300,$$
$$\log. 3 = \cdot 4771213,$$

to find log. 45, log. 450, log. 4·5. State the advantages of the base of the system of logarithms being coincident with the base of our system of arithmetical notation. Why in every system will the logarithm of 1 be 0?

7. Define the sine and cosine of an angle, and trace their changes in sign through four right angles. Also show that

$$\text{Sin. } (A \pm B) = \sin. A \cos. B \pm \cos. A \sin. B.$$

8. Express the area of a triangle in terms of its sides.

(*Oct. 8th.*—Rev. Prof. Heaviside.)

9. Find the rectangular equations—

(1) To a straight line passing through a given point, at right angles to a given straight line.

(2) To the parabola.

1846. *Monday, Oct.* 26*th.*—Examiner,—Mr. JERRARD.

1. Find the difference between $\frac{355}{113}$ and $\frac{157}{50}$, and express $\frac{1}{13}$ as a decimal. What is the test of arithmetical equality?

2. What will £650 amount to in 5 years, at 5 per cent., compound interest?

3. Extract the square root of 9622404, and that of 96224·04; and explain the process.

4. "A ship's company take a prize of £1000, which is to be divided amongst them according to their pay, and to the time during which they have served: now the officers, four in number, have 40*s.* each a month; the midshipmen, 12 in number, have 30*s.* each a month; and they have all served 6 months: the sailors, who are 110 in number, have each 22*s.* a month, and have been on board 3 months. What will be the share of each class?"

5. What is meant by the product of two fractions? When are four quantities said to be proportionals? If a, b, c, d, be proportionals, prove that,

$$(1)\quad a + b : a - b :: c + d : c - d.$$
$$(2)\quad a^n : b^n :: c^n : d^n.$$

6. Investigate an expression for the sum of n terms, (1) of an arithmetic, (2) of a geometric series: and solve the equations,

$$(\alpha)\quad \frac{7x + 2}{3} - \left(x - \frac{x - 6}{5}\right) = 1.$$

$$(\beta)\quad \left.\begin{aligned} 2x + 3y &= 47 \\ 7x - 2y &= 27 \end{aligned}\right\}. \qquad (\gamma)\quad \left.\begin{aligned} xy &= 105 \\ x^2 + y^2 &= 274 \end{aligned}\right\}.$$

Also form an equation the roots of which shall be 2 and 3.

7. Show that the number of combinations of n things, taken r and r together, is equal to the number of combinations of n things taken $n - r$ and $n - r$ together.

8. Assuming the expression for sin. $(A \pm B)$ and cos. $(A \pm B)$, prove that

$$\text{Tan.}\ (A \pm B) = \frac{\tan. A \pm \tan. B}{1 \mp \tan. A \tan. B}.$$

9. Find an expression adapted to logarithmic computation for the area of a triangle in terms of its sides.

(*Oct.* 28*th.*—Rev. Prof. HEAVISIDE.)

10. If $\frac{x}{a} + \frac{y}{b} = 1$ be the equation to a straight line referred to

rectangular co-ordinates, what do a and b represent geometrically? Find the equation to the line drawn from the origin perpendicular to the above line.

11. Find the rectangular equation to the ellipse, the centre being the origin.

1847. *Monday, Oct. 25th.*—Examiner,—Mr. Jerrard

1. What is the present value of £2063 14*s.* due 6 months hence, interest being at 3 per cent.?

2. Extract the square root of 898893·61.

3. Divide $a^n - b^n$ by $a - b$; and reduce

$$\frac{x}{1-x} - \frac{x^2}{1-x^2} - \frac{x^3}{1-x^3}$$

to its most simple equivalent form. What meaning do you assign to a^n, when n is negative or fractional?

4. Give the algebraical definition of proportion, and deduce Euclid's definition for it. Also show that if $a : b :: c : d$, we shall have $a^2 + b^2 : a^2 - b^2 :: c^2 + d^2 : c^2 - d^2$.

5. In how many different ways can seven persons be arranged on seven seats?

6. Solve the equations:

$$(\alpha)\quad \frac{x}{2} - (7x - 40) = \frac{12 - 2x}{11} + 1.$$

$$(\beta)\quad \left.\begin{aligned} 11x + 7y &= 47 \\ 23x - 29y &= 11 \end{aligned}\right\}. \qquad (\gamma)\quad \left.\begin{aligned} x^2 + y^2 &= 146 \\ x + y &= 16 \end{aligned}\right\}.$$

$$(\delta)\quad x^2 - 12x + 32 = 0.$$

7. Show that the logarithm of the pth power of a number is p times the logarithm of the number.

8. Given the sines and cosines of two angles, find the sine and cosine of their sum and difference; and show that

$$\text{Sin. } 36^\circ = \frac{\sqrt{5 - \sqrt{5}}}{2\sqrt{2}}, \text{ cos. } 36^\circ = \frac{\sqrt{5} + 1}{4}.$$

(*Oct. 27th.*—Rev. Prof. Heaviside.)

9. In the equation to a straight line $y = ax + b$, what do a and b represent geometrically? Determine (a) and (b) when the straight line passes through two points whose co-ordinates are (x', y') $\left(0, \frac{y'}{2}\right)$.

10. Find the rectangular equation to the parabola, the vertex being the origin.

1848. *Monday, Oct.* 23*rd.*—Examiner,—Mr. Jerrard.

1. Show that when a number is divisible by 4, its two last figures must be divisible by 4; when by 9, the sum of its digits is divisible by 9; and resolve 1679616 into its factors.

2. Explain the method of extracting the square root of a number. Take as an example 1679616. What is the square root of $2\frac{1}{4}$ to five places of decimals?

3. Find the simple interest of £238 10*s.* 10*d.* for 3 years, at $4\frac{1}{2}$ per cent.

4. Multiply $a + \frac{x}{2a} - \frac{x^2}{4a^2}$ by $x - \frac{a}{2x} + \frac{a^2}{4x^2}$, and reduce $\frac{a+b}{a-b} + \frac{a-b}{a+b}$ to its most simple equivalent form. Explain the rule of signs in the multiplication of algebraic quantities.

5. Solve the equations,

$$(\alpha)\quad \frac{8x}{5} - \left(\frac{x}{7} - 30\right) - 2x = 11. \qquad (\beta)\quad \left.\begin{array}{l} 4x + 5y = 73 \\ 4y - 3x = 15 \end{array}\right\}.$$

$$(\gamma)\quad x^4 + qx^2 + s = 0.$$

The sum of a decreasing arithmetical series is 75, its first term 21, and the common difference 3; to find the number of its terms.

6. Show that in a system of logarithms to the base 10, the logarithms of all numbers which are expressed by the same succession of significant digits, may be found from one opening of the tables.

7. Define the sine and cosine of an angle, and trace their variations through four right angles. Also prove that in a triangle the sides are proportional to the sines of the opposite angles.

8. To express the area of a triangle in terms of the sides.

(*Oct.* 25*th.*—Rev. Prof. Heaviside.)

9. Find the general equation in rectangular co-ordinates to a given circle. And from this deduce the equations (1) when the origin is in the circumference, (2) when it is at the centre of the circle.

1849. *Monday, Oct.* 22*nd.*—Examiner,—Mr. Jerrard.

1. What is the meaning of the term multiplication, as applied to fractions? State the general rule for multiplying any number of fractions together. Also reduce $\frac{36}{1250}$ to a decimal, and divide ·00031 by ·32.

2. A privateer running at the rate of 10 miles an hour, discovers a ship 18 miles off, making way at the rate of 8 miles

an hour: how many miles can the ship run before she will be over taken?

3. Show that in a system of logarithms to the base 10, the logarithms of all numbers between 1 and 10 must be included between 0 and 1. How is the characteristic of a logarithm generally determined? Given log. 2 = ·3010300, log. 3 = ·4771213, to find log. 80, log. 81, log. 360. Point out the utility of logarithms in calculation.

4. Find the product of the four following factors,

$$(a + b),\ (a^2 + a\,b + b^2),\ (a - b),\ (a^2 - a\,b + b^2).$$

What multiplier will render $\sqrt{a} - \sqrt{b}$ rational? Divide $a^{\frac{1}{2}}$ by $a^{\frac{1}{4}}$; and $\frac{a\,c - a\,d}{2\,b}\sqrt{a^2\,x - a\,x^2}$ by $\frac{a}{2\,b}\sqrt{a - x}$.

5. The sum of the progression of uneven numbers, 1, 3, 5, . . . continued to n terms, is equal to n^2. Prove this, and solve the equations:

$$(\alpha)\ \frac{x + 5}{12} - \left(6\,x - 17 - \frac{7\,x - 40}{9}\right) = x - 30.$$

$$(\beta)\ \left.\begin{aligned} x + y - z &= 8 \\ x + z - y &= 9 \\ y + z - x &= 10 \end{aligned}\right\}. \qquad (\gamma)\ \left.\begin{aligned} x^2 + y^2 &= m \\ x + y &= n \end{aligned}\right\}.$$

Show that in every equation of the form $x^2 - a\,x + b = 0$, the two values of x are such that their sum is equal to a, and their product equal to b.

6. Define the sine and cosine of an angle, and trace their variations through four right angles. When the sines and cosines of two angles are given, show how to find the sine and cosine of their sum and difference.

7. Find the expression for the area of a triangle in terms of its sides.

(*Oct.* 24*th.*—Rev. Prof. HEAVISIDE.)

8. Find the equation to a straight line cutting its axes of rectangular co-ordinates in two given points. Find the rectangular equation to the ellipse, the centre being the origin of co-ordinates.

1850. *Monday, Oct.* 28*th.*—Examiner,—Mr. JERRARD.

1. Explain the method of extracting the square root of a number. What is the square root of 2, to five places of decimals?

2. A cistern is filled in 20 minutes by 3 pipes, one of which conveys 10 gallons more, and the other 5 gallons less, than the third per minute. The cistern holds 820 gallons. How much flows through each pipe in a minute?

3. Multiply together $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4}$, and $1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4}$; and divide $a^n - b^n$ by $a - b$. What meaning do you assign to a^n when n is negative or fractional?

Reduce $\frac{x+y}{y} - \frac{2x}{x+y} + \frac{x^3 - x^2 y}{y^3 - x^2 y}$ to its most simple equivalent form, and discuss the case when $x = 1$, $y = 1$.

4. (1) The difference between two numbers is 48, and the arithmetical mean exceeds the geometrical by 18; find the numbers. (2) Also solve the equations:

$$(\alpha)\quad \frac{9x+7}{2} - \left(x - \frac{x-2}{7}\right) = 36.$$

$$(\beta)\quad x^2 + 20x - 341 = 0.$$

$$(\gamma)\quad \left.\begin{aligned} x^2 + y^2 &= 650 \\ x + y &= 36 \end{aligned}\right\}. \qquad (\delta)\quad \left.\begin{aligned} a^x b^y &= c \\ a_1^x b_1^y &= c_1 \end{aligned}\right\}.$$

(3) If x_1, x_2, are the roots of the equation $a x^2 + b x + c = 0$, prove that

$$\frac{x_1}{x_2} + \frac{x_2}{x_1} = \frac{b^2 - 2ac}{ac}.$$

5. From a company of 50 men four are chosen every night to guard. On how many different nights can a different guard be posted; and on how many of these will any particular man be engaged?

6. Define the sine and cosine of an angle, and trace their variations through four right angles. Also prove that in a triangle the sides are proportional to the sines of the opposite angles.

7. Express the area of a triangle in terms of the sides. Take as an example the triangle of which the sides are 3, 4, and 5 units respectively.

(*Oct.* 30*th.*—Rev. Prof. HEAVISIDE.)

8. Find the equation to the straight line perpendicular to the line whose equation is

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Find the rectangular equation to the parabola. Define the latus rectum, and find it in terms of the distance of the focus from the vertex.

1851. *Monday, Oct.* 27*th.*—Examiner,—G. B. JERRARD, Esq.

1. What is the interest of £273 15*s.* for a year, at $3\frac{1}{4}$ per cent.

2. Explain the rule of signs in the multiplication of algebraic quantities; and multiply

[1] $a^3 + a^2 b + a b^2 + b^3$ by $a - b$.
[2] $a^2 + a b + b^2$ by $a^2 - a b + b^2$.

Under what circumstances will $x^2 + a x + b$ be divisible by $x + y$?

3. Insert n arithmetical means between a and b, and find the sum of the series $1 + \frac{1}{2} + \frac{1}{4} +$ *ad infinitum*. Also solve the equations:

$$(\alpha)\ x - \frac{2}{3} - \frac{2x-1}{7} = \frac{4}{21}. \qquad (\beta)\ \left.\begin{array}{l} 2x + 3y = 40 \\ 5x - 12y = 61 \end{array}\right\}.$$

$$(\gamma)\ \left.\begin{array}{l} x + y = 18 \\ x^2 + y^2 = 290 \end{array}\right\}. \qquad (\delta)\ \left.\begin{array}{l} x + y = a \\ x^3 + y^3 = b \end{array}\right\}.$$

How many different arrangements can be formed of the letters in the word *engine*?

4. Define the term *logarithm*. What is meant by the *characteristic* and the *mantissa*? Explain the advantage of choosing 10 as a base of a system of logarithms. Given

Log. 3 = ·4771213, log. 7 = ·8450980,

to find log. 21, log. 210, log. 2·1.

5. Point out the use of the signs + and —, to indicate the directions of lines; and define the sine and cosine of an angle, tracing their variations through four right angles.

6. Given the sines and cosines of two angles, to find the sine and cosine of their sum or difference.

7. Express the area of a triangle in terms of its sides.

(*October 29th.*—Rev. Prof. HEAVISIDE.)

8. Define a conic section, distinguishing the several cases. Is the circle included in your definition of an ellipse? Find the rectangular equation to the ellipse, the centre being the origin.

1852. *Monday, Oct. 25th.*—Examiner,—G. B. JERRARD, Esq.

1. Convert $\frac{1}{7}$ into a recurring decimal; and show that the recurring periods for $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \ldots \frac{6}{7}$ will consist of the same digits. What is the form of those fractions which are convertible into finite decimals?

2. Two detachments of foot being ordered to a station at the distance of 39 miles from their present quarters, began their march at the same time; but one party, by travelling $\frac{1}{4}$ of a mile an hour faster than the other, arrived there an hour sooner; required their rates of marching.

3. Under what circumstances will $x^n + y^n$ be divisible by $x + y$? Is $x^n + y^n$ ever divisible by $x - y$? Reduce

$$\frac{1}{(x+1)^3} - \frac{3}{2(x+1)^2} + \frac{5}{4(x+1)} - \frac{5}{4(x+3)}$$

to its most simple equivalent form.

4. Solve the equations:

$$(\alpha)\ \frac{5x+4}{17} - \left(2 - \frac{x}{2}\right) = 1 + \frac{x}{3}. \qquad (\beta)\ \left.\begin{aligned} 2x + 5y &= 41 \\ 5x - 2y &= 1 \end{aligned}\right\}.$$

$$(\gamma)\ \left.\begin{aligned} x^2 + y^2 &= 170 \\ xy &= 77 \end{aligned}\right\}. \qquad (\delta)\ \left.\begin{aligned} \frac{x+y}{x-y} &= \frac{a}{b} \\ x^2 - y^2 &= c \end{aligned}\right\}.$$

It is required to find four numbers in arithmetical progression, such that if they are increased by 2, 4, 8, and 15 respectively, the sums shall be in geometrical progression.

5. Show that the number of permutations of n things taken r together is $n(n-1)(n-2)\ldots(n-r+1)$.

In how many ways may seven balls be arranged in two divisions, so that the first division may contain three of the balls, the second four?

6. When the sines and cosines of two angles are given, show how to find the sine and cosine of their sum or difference; and thence deduce,

$$\text{Sin. } A + \sin. B = 2 \sin. \frac{A+B}{2} \cos. \frac{A-B}{2},$$

$$\text{Sin. } A - \sin. B = 2 \cos. \frac{A+B}{2} \sin. \frac{A-B}{2}.$$

7. To express the area of a triangle in terms of its sides.

(*Oct. 27th.*—Rev. Prof. HEAVISIDE.)

8. Find the equation to a straight line passing through a given point, and perpendicular to a given straight line. Find the rectangular equation to the hyperbola, making either the centre or the vertex the origin.

1853. *Tuesday, October 25th.*—Examiner,—Mr. JERRARD.

1. Explain the difference between interest and discount. At what rate per cent., simple interest, will £225 amount to £256 10*s.* in 4 years?

Define stock. How much stock can be purchased by the transfer of £2,000 stock from the 3 per cents. at 90 to the $3\frac{1}{2}$ per cents. at 96; and what change will be effected in income by it?

2. A ship having a crew of 26 persons carries provisions for 21 days; after having been at sea for 11 days, they pick up a party

from a wreck, and it is then found that the provisions will be exhausted in the course of five days; find the number of persons taken from the wreck.

3. What is meant by the base of a system of logarithms? Show that, in all systems, the logarithm of the base is 1, and the logarithm of 1 is 0. Point out the advantage of choosing 10 as the base of a system. Given log. 2 = ·3010300, log. 3 = ·4771213, to find log. 360, log. 36, log. 3·6.

4. Find the product of the four binomial factors, $x + a, x + b, x + c, x + d$; and thence deduce the expression when $a = b = c = d$. What multiplier will render $\sqrt{m} - \sqrt{n}$ rational? Divide $a^4 + a^2 b^2 + b^4$ by $a^2 + a b + b^2$; and $a^{\frac{1}{2}}$ by $a^{-\frac{1}{2}}$. Explain the meaning of fractional and negative indices.

5. Solve the equations:

$$(\alpha)\ 5\,x - \frac{2\,x - 18}{3} = 17 - (x - 53). \qquad (\beta)\ \left.\begin{aligned} 7\,x + 11\,y &= 154 \\ 5\,x - \ 3\,y &= \ 34 \end{aligned}\right\}.$$

$$(\gamma)\ \left.\begin{aligned} x^2 + y^2 &= m \\ x - y &= n \end{aligned}\right\}.$$

6. Assuming that

$$\sin. (A + B) = \sin. A \cos. B + \cos. A \sin. B,$$

show how, from the mode of denoting the directions of lines by their signs, to deduce the expressions for sin. $(A - B)$, cos. $(A + B)$, cos. $(A - B)$. What is the numerical value of sin. 30°?

7. Prove that the sides of a triangle are proportional to the sines of the opposite angles.

(*Oct. 27th.*—Rev. Prof. Heaviside.)

8. Find by means of rectangular co-ordinates, the length of the perpendicular let fall from a given point on a given straight line. Define an ellipse, and find the rectangular equation to a given ellipse.

1854. *Monday, Oct. 23rd.*—Examiner,—G. B. Jerrard, Esq.

1. In what time will £350 amount to £402 10*s.* at 3 per cent., simple interest? A person invests £3400 in the 3 per cent. Consols at 95: what amount of stock does he receive, the brokerage being 2*s.* 6*d.* per cent.?

2. The quick-time or step, in marching, being two paces per second, and the length of each pace 28 inches; then at what rate per hour does a troop march on a route, and how long will they be in arriving at a garrison 20 miles distant, allowing a halt of one hour by the way to refresh?

3. Explain the rule of signs in the multiplication of algebraic quantities. What meaning do you assign to a^n, when n is negative or fractional? Under what circumstances will $x^n + y^n$ be divisible by $x + y$? Is $x^n + y^n$ ever divisible by $x - y$?

4. Required the number of permutations of n things taken r and r together.

How many different arrangements can be formed of the letters in the word *infinity?*

5. What number is that, the third part of which exceeds the fourth part by 16? Given

$$x\,y\,(x^2 + y^2) = 3 \ldots\ldots\ldots\ldots A$$
$$x^2 y^2 (x^4 + y^4) = 7 \ldots\ldots\ldots\ldots B$$

to find the values of x and y by a quadratic equation. Also find three numbers in arithmetical progression, such that the sum of their squares shall be 93; and if the first be multiplied by 3, the second by 4, and the third by 5, the sum of the products shall be 66.

6. Given the sines and cosines of two angles, to find the sine and cosine of their sum or difference.

7. Express, in a form adapted to logarithmic computation, the area of a triangle, in terms of the sides.

(*Oct. 25th.*—Rev. Prof. HEAVISIDE.)

8. Assuming the method of representing the locus of points by rectangular co-ordinates, show that the general equation of the first degree between (x) and (y) represents a straight line.

Define the hyperbola, and find a rectangular equation to the hyperbola.

1855. *Wednesday, October 24th.*—Examiner,—Mr. JERRARD.

1. Convert $\frac{1}{7}$ into a recurring decimal; and show that if we take the successive multiples of $\frac{1}{7}$, namely $\frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7} \ldots$ the recurring periods will consist of the same digits. Determine the form of those fractions which are convertible into finite decimals.

2. What is meant by the terms *Involution* and *Evolution?* Find the continued product of the three factors $x + a$, $x + b$, $x + c$. What will the expression thence arising become, when $a = b = c$? Show how to extract the square root of a compound algebraical quantity. Ex. $a^4 - 2\,a^3\,x + 3\,a^2\,x^2 - 2\,a\,x^3 + x^4$. Explain the rule of pointing in the extraction of the square root of a number.

3. Solve the equations:

$$(1)\quad \frac{3x-7}{2} - \frac{19}{93}(5x-4) = \frac{2}{3}. \qquad (2)\quad \left.\begin{aligned} 11x + 7y &= 170 \\ 5x - 3y &= 34 \end{aligned}\right\}.$$

$$(3)\quad \left.\begin{aligned} x + y &= a \\ x^2 + m\,x\,y + y^2 &= b \end{aligned}\right\}. \qquad (4)\quad \left.\begin{aligned} y\,z\,u &= a^3 \\ x\,z\,u &= b^3 \\ x\,y\,u &= c^3 \\ x\,y\,z &= d^3 \end{aligned}\right\}.$$

and show that if α and β be the two roots of a quadratic equation

$$x^2 + \mathrm{a}\,x + \mathrm{b} = 0,$$

then will

$$x^2 + \mathrm{a}\,x + \mathrm{b} = (x - \alpha)\,x - \beta).$$

4. How does it appear that by means of a table of logarithms, *multiplication* may be performed by addition, *division* by subtraction, *involution* by multiplication, and *evolution* by division. Point out the advantage of choosing 10 as the base of a system of logarithms.

5. Explain the meaning of the term angle in trigonometry What is a negative angle? Trace the variations of sin. A, cos. A, in sign and magnitude as A increases from 0° to 360°.

6. Given two sides and the included angle (a, C, b), to solve the triangle.

(*Oct. 25th*,—Rev. Prof. HEAVISIDE.)

7. Find the general equation to the circle referred to rectangular co-ordinates: how is the equation modified when the origin is [1] in the circumference, [2] in the centre of the circle?

Find the rectangular equation to the ellipse, the centre being the origin: what are the axes of the ellipse whose equation is

$$a^2 x^2 + b^2 y^2 = c^4?$$

1856. *Wednesday, Oct. 29th.*—Examiner,—G. B. JERRARD, Esq.

1. A wall was to be built 700 yards long in 29 days. Now, after 12 men had been employed on it for 11 days, it was found that they had completed only 220 yards of the wall. It is required to determine how many men must be added to the former, that the whole number of them may just finish the wall in the time proposed, at the same rate of working.

2. Explain the meaning of fractional and of negative exponents. Also extract the square root of $4\,a^4 + 12\,a^3\,x + 13\,a^2\,x^2 + 6\,a\,x^3 + x^4$, and reduce $3^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$ to quantities which shall have the common exponent $\frac{1}{6}$.

3. Solve the equations:

$$(\alpha)\quad \frac{x}{2} - \frac{x}{3} + \frac{x}{4} = 5. \qquad (\beta)\quad \left.\begin{aligned} 3\,x + 5\,y &= 60 \\ x - 2\,y &= \ \ 9 \end{aligned}\right\}.$$

$$(\gamma)\quad \left.\begin{aligned} x^2 + a\,x\,y + y^2 &= m \\ x + b\,y &= n \end{aligned}\right\}. \qquad (\delta)\quad \left.\begin{aligned} \left(\frac{x}{a}\right)^{\alpha} \cdot \left(\frac{y}{b}\right)^{\beta} &= c \\ \left(\frac{x}{b}\right)^{\beta} \cdot \left(\frac{y}{a}\right)^{\alpha} &= d \end{aligned}\right\}.$$

If in the first of the equations (γ) we take $a = 0$, what will the expressions for x and y become (1) when $b = 1$, (2) when $b = -1$?

Explain the result when $a = 2$, $b = 1$.

4. Insert m arithmetical means between two given numbers; and show that if in any arithmetical progression we insert the same number of arithmetical means between each term and the following one, the new series will also be an arithmetical progression. Does an analogous proposition exist for geometrical progressions?

5. Find the number of permutations of n things taken r together. How many distinct triliteral words can be formed of eight consonants and one vowel, the vowel being always the central letter?

6. Given the sines and cosines of two angles, to find the sine and cosine of their sum or difference.

7. In any plane triangle $A\ B\ C$, express sin. $\frac{A}{2}$, cos. $\frac{A}{2}$ in terms of the sides; and thence deduce the expression for the area.

(*Oct.* 30*th.*—Rev. Prof. HEAVISIDE.)

8. Every equation of the first degree between x and y, is the equation to a straight line. Find the rectangular equation to a straight line passing through a given point, and making a given angle with a given straight line.

Find the rectangular equation to the parabola.

1857. *Wednesday, Oct.* 28*th.*—Examiner,—Mr. JERRARD.

1. If 180 men, in 6 days, working during 10 hours each day, can dig a trench 200 yards long, 3 wide, and 2 deep; in how many days, working during 8 hours each day, will 100 men dig a trench 360 yards long, 4 wide, and 3 deep?

2. Show how to find the highest common divisor of two algebraical expressions. Take, for example,

$$x^3 - b^2 x, \quad x^2 + 2\,b\,x + b^2.$$

3. Solve the equations:

(1) $$\frac{2\,x - 5}{3} - \tfrac{1}{2}(7\,x - 9) = x - 1.$$

(2) $$\left.\begin{aligned} 3\,x + 5\,y &= 34 \\ 17\,x - 7\,y &= 16 \end{aligned}\right\}.$$

and (3) divide the number 60 into two such parts that their product shall be to the sum of their squares in the ratio of 2 to 5.

4. If $a : b :: c : d$,

prove that $m\,a \pm n\,b : p\,a \pm q\,b :: m\,c \pm n\,d : p\,c \pm q\,d.$

How does it appear that Euclid's definition of proportion and the algebraical follow each from the other?

5. Define the sine and cosine of an angle; and trace the variations of sin. A, cos. A, in sign and magnitude as A increases from $0°$ to $360°$.

6. Given two angles and the side between them ($A\ C\ b$), to solve the triangle.

(*Oct.* 29*th.*—Rev. Prof. HEAVISIDE.)

7. A straight line cuts the axes of co-ordinates at given distances from the origin, find the equation to the straight line, in terms of those distances.

If $y = 2x + 3$ be the equation to a straight line, find the equation to the line perpendicular to it from the origin: find also the length of this perpendicular.

Find a rectangular equation to the hyperbola.

1858. *Wednesday, Oct.* 27*th.*—Examiner,—G. B. JERRARD, Esq.

1. A railway train after travelling for one hour meets with an accident which delays it one hour, after which it proceeds at three-fifths of its former rate, and arrives at the terminus three hours behind time; had the accident occurred 50 miles further on, the train would have arrived 1 hour 20 minutes sooner. Required the length of the line.

2. Multiply together (1) $a + x$, $b + x$, and $c + x$; and resolve (2) $x^6 - a^6$ into four factors.

Explain the meaning of a^n when n is (3) fractional and (4) negative.

3. (1) Define the base of a system of logarithms. (2) What is meant by the characteristic and the mantissa? (3) Show that in the common system the characteristic of the logarithm of any number can be determined by inspection.

Given log. 2 = ·301030, log. 3 = ·477121, find (4) log. 24, (5) log. 5·4, and (6) log. ·006.

4. Solve the equations:

$$(\alpha)\quad \frac{x+6}{2} - \frac{x-7}{3} = 2x - 13.$$

$$(\beta)\quad \left.\begin{aligned} 7x + 11y &= 57 \\ 13x - 21y &= 23 \end{aligned}\right\}.$$

$$(\gamma)\quad \left.\begin{aligned} x^2 + xy + y^2 &= 7 \\ x - y &= 3 \end{aligned}\right\}.$$

(1) What condition must be fulfilled in order that the two roots of the quadratic equation

$$a x^2 + b x + c = 0$$

may be equal? (2) How does it appear that a quadratic equation cannot have more than two roots?

5. Prove that

$$\sin.(A + B) = \sin. A \cos. B + \cos. A \sin. B \cdot$$

and deduce formulæ for sin. $(A - B)$, cos. $(A + B)$, and cos. $(A - B)$.

6. To find an expression for the area of a triangle in terms of the sides, in a form adapted to logarithmic computation.

(*Oct.* 28*th.*—The Rev. J. W. L. HEAVISIDE.)

7. What is the general form of the equation of the first degree between two variables? show that it represents a straight line in co-ordinate geometry.

Find the rectangular equation to a straight line in terms of the perpendicular upon it from the origin, and the angle which that perpendicular makes with one of the axes.

Find the general rectangular equation to a circle.

1859. *Tuesday, July* 19*th.*—Examiner,—E. J. ROUTH, Esq., M.A.

1. A mixture of black and green tea is sold so as to gain 4 per cent. on the outlay. If sold separately at the same price per lb., the gains would have been 5 and 3 per cent. respectively. In what proportion were the two kinds of tea mixed?

2. Simplify the expressions:

$$(1)\quad \frac{1}{13} \cdot \frac{\frac{2}{8} + \frac{4}{7}}{\frac{2}{8} - \frac{4}{7}} - \left(\frac{3}{4} + \frac{5}{6} - 2\left(\frac{3}{8} - \frac{1}{12}\right)\right),$$

$$(2)\quad \frac{1}{\sqrt{1+x} + \sqrt{x}} + \frac{1}{\sqrt{1+x} - \sqrt{x}},$$

and (3) find the value of $x^2 + a\,x + b$ when

$$x = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b}.$$

3. Define the greatest common measure of two arithmetical, and also of two algebraical quantities. If H be the greatest common measure of two algebraical quantities P and Q, will it still be the greatest common measure when numerical quantities are given to the letters in H, P, and Q?

4. Give both the ordinary algebraic definition, and also Euclid's definition of proportion. Show how to deduce the former from the latter.

If $a : b :: c : d$, prove that

$$a\,d^3 + b\,c^3 : a^3\,d + b^3\,c :: c^2\,b^4 + a^2\,d^4 : (a\,b + c\,d)\,a\,b^3.$$

If b and d are very nearly equal, show that this ratio is very nearly equal to that of $3\,d - 2\,b : d$.

5. Show how to find the number of combinations of n things taken κ together.

The total number of combinations of $p + q$ things, of which p are of one sort and q of another, taking them first one, then two, three, &c., together, is $p + q + p\,q$.

6. Define (1) a geometrical progression. (2) Is it possible for three numbers to be both in arithmetical and geometrical progression?

(3) If s be the sum of a geometrical progression whose first term is a and last term l, and s' the sum of the reciprocals of the terms of the same series, then prove that $\frac{s}{s'} = a\,l$.

7. Investigate a rule to find the compound interest of any sum for n months at a given rate per cent. per annum.

(2) A person spends every year a certain fraction of his income, and continually adds the remainder to his capital; what fraction ought this to be, that, after a given number of years, his whole income may be increased n times?

8. Show that the equation

$$\frac{x+6}{x-1} + \frac{x-6}{x+1} = 2\,\frac{x^2-6}{x^2-1}$$

does not admit of any solution except $x = \infty$.

Solve

$$\left.\begin{aligned} 4\,x^2 + 7\,x\,y + 2\,y^2 &= 13 \\ 5\,x\,y + 7\,y^2 &= 12 \end{aligned}\right\}.$$

9. Define the characteristic of a logarithm. What are the characteristics of the logarithms of 234 and ·0067, the base of the system of logarithms being 7? What are the comparative advantages of the common and Napierian system of logarithms?

Find the value of $\overline{\cdot 001}|^{\cdot 001}$, having given

log. 9·9328 = ·9960323,
log. 9·9329 = ·9970367.

(*July* 19*th*.—Rev. Prof. Heaviside.)

10. Define an ellipse; find a rectangular equation to the ellipse. If the equation is referred to the centre as origin, show how to refer it to the vertex; and if the equation is referred to the vertex as origin, show how to refer it to the centre.

11. How is an angle measured in trigonometry? What is the cosine of an angle? For what angle less than 90° is the sine equal to the cosine?

Express the numerical value (1) of a fifth part of two right angles, (2) of the complement of 62° 15′, (3) of the angle of a regular pentagon.

12. Prove $\tan.\,\overline{A+B} = \dfrac{\tan.\,A + \tan.\,B}{1 - \tan.\,A\,.\,\tan.\,B}$.

Account for the value of tan. $A + B$ given by this formula, if $A = B = 45°$.

13. In every triangle the sines of the angles are proportional to the sides opposite to them.

Find the area of the triangle whose sides are 30, 40, 50 feet.

14. Given two angles and a side of a plane triangle; solve the triangle.

Given two sides and an angle opposite to one of them; show how to solve the triangle. Why is the solution in the latter case ambiguous?

Ex. Given $A = 52° \, 15' \, 35''$, $a = 500$, $C = 88° \, 30'$.
Find c, B.

$$\log. \sin. 88° \, 30' = 9{\cdot}9998512.$$
$$\log. \sin. 52° \, 15' \, 35'' = 9{\cdot}8980630.$$
$$\log._{10} 5 = {\cdot}6989700.$$
$$\log._{10} 6{\cdot}3206 = {\cdot}8007582.$$

How do the tabulated logarithms of the sines of angles differ from the logarithms of the sines calculated to a base 10?

1859. *Wednesday, Oct. 26th.*—Examiner,—Rev. J. W. L. Heaviside.

1. A mass of lead-ore, weighing 800 grains Troy, was found to contain 6 grains of silver: what is the value of the silver in one ton of the ore, at the rate of 5*s*. an ounce Troy, it being given that one pound avoirdupois contains 7000 grains Troy?

2. Prove

$$(1) \quad \frac{2x^3 - 13x + 15}{3x^3 + 9x^2 - 5x - 15} = \frac{2x^2 - 6x + 5}{3x^2 - 5}.$$

$$(2) \quad \frac{x^2 - xy + y^2}{x^2 y^2}\left\{\frac{1}{x} + \frac{1}{y}\right\} - \frac{y^2 - yz + z^2}{y^2 z^2}\left\{\frac{1}{y} + \frac{1}{z}\right\} = \frac{x^2 + xz + z^2}{x^2 z^2}\left\{\frac{1}{x} - \frac{1}{z}\right\}.$$

3. Solve the following equations:

$$(1) \quad 3x - \frac{169 - 3x}{x} = 29. \qquad (2) \quad (x - a)(x - b) = (c - a)(c - b).$$

Find three numbers, in geometrical progression, the product of which is 729, and the sum of the squares 819.

4. Find the present value of an annuity, to be continued for any given number of years, at a given rate per cent., allowing compound interest. What does the expression for the present value become, when the annuity is perpetual? How many years' purchase must be paid for a freehold estate returning a given rent, allowing interest at five per cent.?

5. Explain why $\log._{10} 6{\cdot}25$, $\log._{10} {\cdot}000625$, $\log._{10} \frac{1}{16}$, have each the same mantissa; write down the logarithms of these numbers, it being given that $\log._{10} 2 = {\cdot}301030$. Account for the registered logarithms of the sines and cosines of angles being all positive. Given

$$\log.\tan. 46^\circ\, 17' = 10{\cdot}019462, \text{ find } \log.\cot. 46^\circ\, 17'.$$

6. Prove $\sin.(A + B) = \sin. A \cos. B + \cos. A \sin. B$, and deduce from it the expression for $\cos.(A - B)$. Could $\sin.(A + B)$ be expressed in terms of $\sin. A$ and $\sin. B$ only?

7. If two sides and the included angle of a triangle are given, show how to solve the triangle.

Ex. The two sides are 345, 174 feet respectively, and the included angle is $37^\circ\, 20'$; find the remaining angles of the triangle.

$$\log._{10} 5{\cdot}19 = {\cdot}715167, \quad \log.\tan. 71^\circ\, 20' = 10{\cdot}471298.$$
$$\log._{10} 1{\cdot}71 = {\cdot}232996, \quad \log.\tan. 44^\circ\, 17' = 9{\cdot}989127.$$

(*October* 27*th.*—E. J. ROUTH, Esq.)

8. If $y = a\,x + b$ be the equation to a straight line referred to rectangular co-ordinates, *state* the geometrical meanings of the quantities a and b.

Find the equation to the straight line which is drawn perpendicular to a given straight line from a given point.

State how a cone must be cut by a plane, that the section may be a parabola.

1860. *Tuesday, July* 17*th.*—Examiners,—W. H. BESANT, Esq., M.A., and E. J. ROUTH, Esq., M.A.

1. If money invested in the 3 per cent. consols. give exactly 3 per cent., after the payment of one shilling in the pound income tax, find the price of the consols., allowing $\frac{1}{8}$th per cent. to the broker for the purchase.

A person invests £3,000 in the 3 per cents., at $94\frac{7}{8}$, and £2,000 in 6 per cent. Canada bonds, at $112\frac{1}{8}$; allowing in each case $\frac{1}{8}$th per cent. on the purchase to the broker, find the average per-centage obtained on the £5,000 invested.

2. Simplify the expression:

$$(1) \quad \left\{\frac{11\frac{3}{4} - 10\frac{1}{2}}{11\frac{3}{4} + 10\frac{1}{2}} \div \frac{10\frac{3}{8} + 11\frac{1}{8}}{10\frac{3}{8} - 9\frac{1}{8}}\right\} \times \frac{\frac{2}{7} + \frac{3}{11}}{\frac{2}{7} - \frac{3}{11}}.$$

(2) Divide $\cdot00002\dot{4}37\dot{4}$ by $\cdot00000\dot{2}43\dot{5}$, and (3) find the square roots of 9947716, and (4) $\cdot\dot{0}4938271\dot{6}$.

3. Divide

$$a^6 + a^5 b + a^4 b^2 + a^3 b^2 c + a^3 b^3 c + a^3 b c^2 + a^3 c^3 + a^2 b^3 c + a^2 b c^3$$
$$+ a b^3 c^2 + b^3 c^3 + b c^5 \text{ by } a^3 + a^2 b + b c^2;$$

and if n be a positive integer, prove that $x^{2n+1} + y^{2n+1}$ is always divisible by $x + y$.

Also simplify the expression:

$$\left\{\sqrt{\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}}+\sqrt{\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}}\right\}$$
$$+\left\{\frac{\sqrt{3+2x}+\sqrt{3-2x}}{\sqrt{3+2x}-\sqrt{3-2x}}-\frac{\sqrt{3+2x}-\sqrt{3-2x}}{\sqrt{3+2x}+\sqrt{3-2x}}\right\}.$$

4. Define ratio and proportion; and deduce Euclid's definition V., Book V., from the algebraical definition. If $a:b=c:d=e:f$, prove that each of these ratios

$$\sqrt{\frac{a^2c^2}{e^2}+\frac{a^2e^2}{c^2}+\frac{c^2e^2}{a^2}}:\sqrt{\frac{b^2d^2}{f^2}+\frac{b^2f^2}{d^2}+\frac{d^2f^2}{b^2}}$$
$$=a^3df+c^3bf+e^3bd:b^3ce+d^3ae+f^3ac.$$

5. Find the number of permutations of n things r together, and deduce the number of combinations of n things r together.

If n represent the number of combinations of n things taken r together, prove by general reasoning that

$$(n)_r+(n)_{r-1}=(n+1)_r \text{ and}$$
$$(m+n)_r=(m)_r+(m)_{r-1}(n)_1+(m)_{r-2}(n)_2+\dots+(n)_r.$$

6. Find the sum of a series of quantities in geometrical progression, having given the first term and the common ratio.

If the sum of the nth and $(2n)$th terms of a geometrical progression be given, and also the sum of the $(2n)$th and $(3n)$th terms, find the first term and the common ratio.

Find the sum of n terms of the series of which the rth term is

$$2r+3+2\times 3^r.$$

7. (1) Explain what are meant by discount and present value; and (2) find expressions for the discount and present value of a sum due at the end of a given time, reckoning compound interest.

(3) A given sum is to be invested in an annuity such that each annual payment is one-third of the preceding, to continue for n years; reckoning compound interest at a given rate, find the payment for the first year.

8. Solve the equation:

$$6x^2-17ax+4bx+12a^2-7ab-10b^2=0.$$

A number of men are first formed into a solid square, and afterwards into a hollow square three deep; the front presented in the latter formation has 75 men more than the front in the solid square: determine the number of men.

9. Define a logarithm, and find the logarithm of $a^{\frac{p}{q}}$ to the base $a^{\frac{m}{n}}$.

Show that the characteristic of the logarithm of any number or

decimal can be determined by inspection when the base is 10 and prove the formula

$$\log._a N = \log._a 10 \cdot \log._{10} N.$$

Having given $\log._{10} 2 = \cdot 301030$

$\log._{10} 3 = \cdot 477121,$

find $\log._{10,000} \cdot 0000432.$

(*July 17th.—Afternoon.—Trigonometry and Conics.*)

10. Define the sine of an angle.

Prove that $\sin. A = \sin. (180° - A)$; and discuss the case in which A is greater than two right angles.

Write down in one formula all the angles which have $\frac{1}{2}$ for their sine. Solve the equation

$$1 + \sin. 2\,\theta = \cos. \theta - \sin. \theta.$$

11. Prove that in any triangle

$$\cos. A = \frac{b^2 + c^2 - a^2}{2\,b\,c}.$$

Why is this formula not fit to determine the angles of a triangle when the three sides are given? What formula would you employ in such a case?

12. Prove that if A and B be any two angles,

$$\cos. (A - B) = \cos. A \,.\, \cos. B + \sin. A \,.\, \sin. B.$$

Prove also that

$$\sin. 2\,A \,.\, \sin. A = \cos. A - \cos. A \,.\, \cos. 2\,A;$$

and express $\cot. 2\,A$ in terms of $\cot. A$.

13. Find the equation to the perpendicular drawn from a given point on the straight line whose equation is

$$\frac{x}{a} + \frac{y}{b} = 1.$$

What angles do the straight lines $x - \kappa\, y = 0$ and $y + \kappa\, x = 0$ make with each other?

14. Find the equation to the circle passing through the origin, and having its centre on the axis of x, and the radius of which is equal to a.

Interpret each of the equations

$$x^2 + y^2 = 0 \text{ and } x^2 - y^2 = 0.$$

A point moves so that the sum of the squares of its distances from the three angles of a triangle is constant. Prove that it moves along the circumference of a circle.

15. Investigate the equation to a parabola, $y^2 = 4\,a\,x$.

Explain the geometrical meaning of the constant a. Trace the form of the curve from its equation.

If $T\,A$, $T\,B$ be two tangents to a parabola cutting each other in the principal diameter, then if a third tangent cut them in P and Q, prove that $S\,P = S\,Q$ where S is the focus.

1860. *July 17th. First B. Sc. Examination.*
Examiners,—W. H. BESANT, Esq., M.A., and E. J. ROUTH, Esq., M.A.

☞ *The only separate Examination Paper for this degree. After this date, "the Papers and Examiners are the same as those on the same day at the First B.A. Examination."*

1. A contractor undertook to build a house in 21 days, and engaged 15 men to do the work. But after 10 days he found it necessary to engage 10 men more, and then he accomplished the work one day too soon. How many days behindhand would he have been if he had not engaged the 10 additional men?

2. Cube $(1 - x + 2\,x^2)$, and determine the first six terms of the square of $1 + x + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \frac{x^4}{1\,.\,2\,.\,3\,.\,4} + \ldots\ldots$

Simplify $\frac{2 + \sqrt{3}}{1 + \sqrt{3}}$ and $\sqrt{\dfrac{\dfrac{-1}{1 - x^2} + 1}{\dfrac{1}{1 - \dfrac{1}{x^2}} - 1}}$

and extract the square root of $5 + \sqrt{6} + \sqrt{10} + \sqrt{15}$.

3. Solve the quadratic equation $a\,x^2 + b\,x + c = 0$, and show how to make the equation whose roots shall be α and β.

If α and β be the roots of the equation $a\,x^2 + b\,x + c = 0$, prove that $\frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{b\,(3\,a\,c - b^2)}{c^3}$.

Solve the simultaneous equations

$$\left.\begin{aligned} 2\,x^2 + 3\,x\,y - 4\,y^2 &= 10, \\ x^2 - 2\,x\,y + 3\,y^2 &= 3. \end{aligned}\right\}$$

4. Define a logarithm, and prove that

$$\log.\,(M\,N) = \log.\,M + \log.\,N.$$

What is the modulus of a system of logarithms? State the advantages of choosing 10 as the base of the system.

Explain generally how you would find, by the help of a table of proportional parts, the logarithm of a number consisting of six figures, from a table giving the logarithms of all numbers of five figures.

5. Define the sine and cosine of an angle. Find an expression for all the angles which have tan. α for their tangent.

Prove that if $A + B + C = 180$,

$$\frac{\sin. A + \sin. B - \sin. C}{\sin. A + \sin. B + \sin. C} = \tan. \frac{A}{2} . \tan. \frac{B}{2}.$$

6. In any triangle, prove that

(1) $\frac{\sin. A}{\sin. B} = \frac{a}{b}.$

(2) $\frac{\tan. \frac{A}{2}}{\tan. \frac{B}{2}} = \frac{s - b}{s - a}.$ Where $s = \frac{a + b + c}{2}.$

Explain how you would proceed to find the distance from a given point to any object on the other side of a river, supposing that you cannot cross the river, and that you have no instrument for measuring angles.

7. Find the equation to the straight line drawn from the given point (h, k) perpendicular to the given straight line $a x + b y + c = 0$.

Find also the equation to the system of circles passing through this given point, and touching this straight line. Prove, in any way, that the centres of these circles lie on a parabola.

8. A point P moves so that the sum of its distances from two given points, A and B, is constant. Find the locus of P.

Let S, H, be the foci of an ellipse, and A the extremity of the major axis. Then if P be any point on the ellipse, prove, in any way, that the bisectors of the angles $P S A$, $P H A$ meet in the tangent at P.

(*July* 17*th*.—*Afternoon.*)

9. Investigate the equation to the parabola, $y^2 = 4 a x$.

Explain the geometrical meaning of the constant a. Trace the form of the curve from its equation.

If $T A$, $T B$, be two tangents to a parabola, cutting each other in the principal diameter, then if a third tangent cut them in P and Q, prove that $S P = S Q$, where S is the focus.

1861. *July* 16*th*.—Examiners,—W. H. Besant, Esq., M.A., and E. J. Routh, Esq., M.A.

1. A person invests in £10 railway shares when they are at a premium of ten shillings. At the end of a year he receives a guinea per share. What interest does he get?

If 81 bushels of wheat are consumed by 56 men in 5 days, how long will 16 men take to consume 28 bushels?

2. State and prove the rule for the determination of the fraction equivalent to a given recurring decimal.

Simplify $\frac{2\frac{1}{2} + 1\frac{1}{3}}{2\frac{1}{2} - 1\frac{1}{3}} \times \frac{1\frac{7}{12} - 1}{\frac{1}{4} + \frac{5}{3}}$; and divide $\cdot000741$ by $2\cdot47$.

3. Find the square root of $x^4 - 4x^3 + 2x^2 + 4x + 1$, and simplify

$$\sqrt{\frac{1 - x - \sqrt{2x + x^2}}{1 - x + \sqrt{2x + x^2}}} + \sqrt{\frac{1 - x + \sqrt{2x + x^2}}{1 - x - \sqrt{2x + x^2}}}.$$

4. If $\frac{a}{b}$ and $\frac{c}{d}$ be two equal fractions, prove that they are each $= \frac{a + c}{b + d}$.

If x vary as y, prove that $x^2 + y^2$ will vary as $x^2 - y^2$.

5. Explain the difference between combinations and permutations. Find the number of permutations of n things taken r together.

6. Find the sum of an arithmetical progression, the first and last terms and the number of terms being given.

Sum the following series to n terms—

$$2 + 3\tfrac{1}{2} + 5 + \ldots\ldots$$
$$2 + 3\tfrac{1}{2} + 6\tfrac{1}{8} + \ldots\ldots$$

Find the sum of n terms of the series whose nth term is $2(2^{n-1} + n) + 5$.

7. Solve the equations

$$2x^2 + 11x + 15 = 0.$$
$$bx^2 + 3ax + b - a = ax^2 + 3bx + a - b$$
$$\left.\begin{aligned} x^2 + y^2 &= a^2 \\ x + y &= b \end{aligned}\right\}.$$

8. Find a number of two digits, which is three times the sum of its digits, and such that the difference between the digits is 5.

9. Define a logarithm, and prove that

$$\log. xy = \log. x + \log. y.$$

In what respects is the ordinary system of logarithms to base 10 more convenient, and in what respects less convenient than the Napierian system?

Having given log. 2 = ·301030, and log. 3 = ·477121, find log. ·0144.

Find the characteristic of log. 3·2 to base 5.

(July 16th.—Afternoon.—Trigonometry and Conics.)

10. Define the tangent of an angle, and prove that

$$\tan. A = -\tan. (180^\circ - A).$$

Find an expression for all the angles which have the same tangent as a given angle A, and solve the equation

$$3 \tan.^4 \theta - 10 \tan.^2 \theta + 3 = 0.$$

11. If A and B be each less than 90°, prove that

$$\sin. (A - B) = \sin. A \cos. B - \cos. A \sin. B;$$

also prove the formula

$$\cot. A - 2 \cot. 2 A = \tan. A,$$

$$\frac{\sin. \left(\frac{\pi}{3} + \theta\right) + \cos. \left(\frac{5}{6}\pi - \theta\right)}{\sin. \left(\frac{5}{6}\pi - \theta\right) + \cos. \left(\frac{\pi}{3} + \theta\right)} = \tan. \theta.$$

12. If two sides and the included angle of a triangle be given, show how to solve the triangle.

The angle A of a triangle $A\,B\,C$ is 60°, and the side $A\,C$ is twice the side $A\,B$; find the angles B and C.

13. Explain what is meant by the locus of an equation in x and y when x and y are the co-ordinates of a point referred to fixed axes.

Find the loci of the equations

(1) $x = 3y$. (2) $(x^2 - a^2)^2 (x^2 - b^2)^2 + c^4 (y^2 - a^2)^2 = 0$.

Also define the equation to a curve, and find the equation to a straight line.

14. If α, β be the co-ordinates of the centre, and C the radius of a circle, find its equation, the axes being rectangular.

Find the conditions that the circle may cut off from the axes chords of which the lengths are respectively a and b.

15. Define an ellipse, and find its equation referred to the centre and axes.

If the axes of an ellipse be given in direction, and if the ratio of their lengths be also given, through how many points can the curve be drawn?

16. Trace the form of the curve $y^2 - x^2 = a^2$, and find the equation to the tangent at the point (x, y).

Find x and y when the tangent cuts off a given area (a^2) from the axes.

1862. *July 22nd.—Morning*, 10 *to* 1.

Examiners,—W. H. Besant, Esq., M.A., and E. J. Routh, Esq., M.A.

1. Find the amount of £1,000 placed at simple interest for $11\frac{3}{4}$ years at $3\frac{1}{4}$ per cent.

The sum of £9,040 16*s*. is placed in the $3\frac{1}{2}$ per cents. at 94; find the income obtained, allowing on the stock purchased $\frac{1}{8}$th per cent. to the broker, and $\frac{1}{20}$th per cent. for other expenses.

2. Define a recurring decimal, and show how to reduce recurring decimals to ordinary fractions.

Express as a fraction $\cdot 2\dot{0}012\dot{3}$, and express as a recurring decimal $\cdot 0\dot{1}\dot{2} \div \cdot 00\dot{1}3\dot{2}$.

3. Multiply together $x^3 - 3x^2 + 12x - 1$, $4x^3 - x + 1$, and $x^3 + x^2$.

Extract the square root of

$$x^6 - 6x^4 + 4x^3 + 9x^2 - 12x + 4;$$

and resolve into quadratic factors the expression

$$(5x^2 - 11x + 12)^2 - (4x^2 - 15x + 6)^2.$$

4. Define variation.

If $A \propto B$ when C is constant, and if $A \propto \frac{1}{C}$ when B is constant, prove that generally $A \propto \frac{B}{C}$.

If $x \propto y$, prove that $x^2 + y^2 \propto xy$. Also, if $x^3 + \frac{1}{y^3} \propto x^3 - \frac{1}{y^3}$, prove that xy is constant.

5. Having given the number of permutations, find the number of combinations, of n things r together.

In how many ways can 8 coins be placed in a row on a table, and in how many of these will a particular coin be at an end?

How many different parcels of 4 can be made of these coins, and in how many of these will a particular coin occur?

6. Find the sum of n terms of a given geometric series, and also the sum to infinity when the common ratio is less than unity, explaining what is meant by the sum of an infinite series.

Sum the series

$$a - \frac{a^2}{r^2} + \frac{a^3}{r^4} - \dots\dots$$

to n terms and to infinity, a being less than r^2.

Also find the sum of n terms of the series of which the rth term is $2r + 2^r(1 + 2^r + 4^r)$.

7. Solve the equations

$$\overline{x+b}\;\overline{x+c} + \overline{x+c}\;\overline{x+a} = \overline{2x+a}\;\overline{x+b};$$

$$\left\{\begin{array}{l} x + y = a \\ ax + by = ab \end{array}\right\}; \qquad \left\{\begin{array}{l} x^2 + 4y^2 = 116 \\ xy = 20 \end{array}\right\};$$

$$x^4 + x^2 - 4\sqrt{x^4 + x^2 - 25} = 550.$$

8. A certain number has two digits, the sum of the squares of which is 130; and if the order of the digits be changed, the number is increased by 18. Find the number.

9. Define a logarithm, and find that of 81 to the base $\sqrt{3}$, and of x to the base $\frac{1}{\sqrt{x}}$.

Prove that $\log._a \frac{x}{y} = \log._a x - \log._a y$, and that $\log._a x = \log._a b \,.\, \log._b x$.

Having given $\log._{10} 2 = \cdot 301030$, find $\log._{10} 32$ and $\log._{100} 32$.

10. Obtain an algebraic expression for the simple interest of £P for n years, taking r as the interest of £1 for a year; and also for the present value, reckoning simple interest, of an annuity of £P to commence n years hence and to continue for n years.

(*July* 22.—*Afternoon*, 3 *to* 6.)

11. Define the complement of an angle. Prove that the sine of any angle is the cosine of its complement; and discuss the case in which the angle is greater than a right angle.

Find all the values of x which satisfy the equations:

$$(1)\ \sin. 2\,x = \cos. 3\,x;$$
$$(2)\ 1 + \sin. x = 2 \cos.^2 x.$$

12. Prove that when A lies between 90° and 180°, and B between 0° and 90°,

$$\sin. (A - B) = \sin. A \cos. B - \cos. A \sin. B.$$

Establish the equalities:

$$\sin. 3\,A \sin. A = \sin.^2 2\,A - \sin.^2 A;$$
$$\frac{1 + \cos. 2\,A}{\sin. 2\,A} = \cot. A.$$

13. Show how to solve a triangle when two of its sides and the included angle are given.

Two sides of a triangle are 6 and 8 feet, and the area is 12 square feet: find the third side.

14. Show that the equation

$$x \cos. a + y \sin. a - p = 0$$

represents a straight line and a straight line only.

What is the geometrical meaning of the constants a and p?

Give diagrams of the loci of the equations

$$x^2\,y = 0, \quad x^2 + y^2 = 0, \quad x - y = 4.$$

15. Show how to determine the position and magnitude of the curve represented by the equation

$$A\,x^2 + A\,y^2 + B\,x + C\,y + E = 0.$$

Find the equations to the two straight lines joining the origin to the points of intersection of the two curves

$$\left.\begin{array}{l} x^2 + y^2 = a^2, \\ y = b\,x + c \end{array}\right\}.$$

16. Find the equation to the tangent to the conic

$$A\,x^2 + B\,y^2 = C$$

at any point (x, y_1).

If the tangent at any point P cut the axes of the curve, produced if necessary, in T and T', and if C be the centre of the curve, prove that the area of the triangle $T\,C\,T'$ varies inversely as the area of the triangle $P\,C\,N$, where $P\,N$ is the ordinate of P.

1863. *Tuesday, July* 21.—*Morning,* 10 *to* 1.
Examiners,—W. H. Besant, Esq., M.A., and E. J. Routh, Esq., M.A.

1. Find the rate per cent. at which £1,000 must be laid out at simple interest to become £1,100 in 5 years.

The three per cents. being at 93, determine the interest obtained for money thus invested.

2. Define a decimal fraction; and taking ·237 as an example, show from your definition that $\cdot 237 = \dfrac{237}{1000}$.

Divide 3085·5 by ·00051; and reduce to a vulgar fraction the recurring decimal 2·3017017017 ......

3. Extract the square root of $x^4 - 6\,x^3 = 7\,x^2 + 6\,x + 1$; and simplify

$$\frac{a^8 - 2\,a^4\,x^4 + x^8}{a^6\,x^2 + a^2\,x^6} \div \frac{a^4 - x^4}{a^4 + x^4}.$$

Find also the factors common to the two expressions

$x^4 + 6\,x^3 - 8\,x^2 + 1$, and $x^6 + 7\,x^5 - 3\,x^4 - 3\,x - 2$.

4. When are four quantities said to be proportionals?

If $a : b :: c : d$, prove that

$$\frac{a^3 + 3\,a^2\,b + b^3}{c^3 + 3\,c^2\,d + d^3} = \frac{a^3 + b^3}{c^3 + d^3}.$$

Show how to deduce the algebraical definition of proportionals from that given by Euclid; and consider the case in which the quantities are incommensurable.

5. Find the number of permutations of n things taken r together.

How many words of four consonants and one vowel can be formed from seven consonants and three vowels, the vowel being always in the middle place?

6. Given the first and last terms of an arithmetical progression of n terms, find the series.

Sum the following series:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots\ldots \text{ to infinity;}$$

$$1^2 + 2^2 + 4^2 + 8^2 + \ldots\ldots \text{ to } n \text{ terms.}$$

7. Solve the equation:

$$\frac{1}{x - a} - \frac{1}{x + a} = \frac{1}{x - b} - \frac{1}{x + b}.$$

What is the meaning of your result when $a = b$?

Solve also:

$$\left.\begin{aligned} x^2 - y^2 &= a^2 \\ x - y &= b \end{aligned}\right\},$$

$$\text{and } 2\,x\sqrt{x^2 + x - 1} = 2\,x^2 - 5\,x + 2.$$

8. Two partners, A and B, gained £17 by trade. A's money was in trade one year and a half, and he received for his principal and interest £39. B's money was in trade two years, and he began with £45. What money did A begin with?

9. Define the logarithm of a number to any base. Can any number be taken as the base?

Prove that log. $(M\,N) =$ log. $M +$ log. N.

Find the characteristic of log. ·0003 to base 8.

(July 21st.—Afternoon 3 to 6.—Trigonometry and Conics.)

10. Define the sine of an angle, and prove that

$$\sin.\,\theta = \sin.\,(\pi - \theta).$$

Trace the changes in sign of the expressions sin. $4\,\theta$ and sin. (sin. θ) as θ changes from 0 to π; and solve the equations

$$(1) \quad \sin.\,3\,\theta = \sin.\,4\,\theta;$$
$$(2) \quad 2\sin.\,(\sin.\,\theta) = 1.$$

11. If $A + B$ be less than 90°, prove that

$$\sin.\,\overline{A + B} = \sin.\,A\,.\cos.\,B + \cos.\,A\,.\sin.\,B;$$

and prove the formulæ

$$4\sin.\,3\,A\,.\sin.\,5\,A\,.\sin.\,7\,A = \sin.\,A + \sin.\,5\,A + \sin.\,9\,A - \sin.\,15\,A;$$

$$\frac{\tan.\,\overline{\theta + \alpha} + \tan.\,\overline{\theta - \alpha}}{\cot.\,\overline{\theta + \alpha} + \cot.\,\overline{\theta - \alpha}} = \tan.\,\overline{\theta + \alpha}\,.\tan.\,\overline{\theta - \alpha}\,.$$

12. Find an expression for the cosine of an angle of a triangle in terms of the sides.

Having given the sides of a triangle, investigate a convenient formula for determining its angles by logarithmic computation.

If one of the angles be very small, will your formula determine the angle accurately? and if not, what modification would you employ?

13. Define the equation to a curve; and find the equation to a straight line in the form

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Find the equation to a straight line which passes through a given point, and cuts off a given area from the co-ordinate axes, determining the condition that this may be possible.

14. Interpret the equations

$$(1) \quad \overline{x-a}|^2 + \overline{y-b}|^2 = c^2;$$
$$(2) \quad x^2 + y^2 + a^2 + b^2 = 2\,a\,x + 2\,b\,y.$$

Find the equation to the circle passing through the origin and the points $(a, b,)$ (b, a); and determine the lengths of the chords it cuts from the axes.

15. Define a parabola, and find its equation referred to its axis and the tangent at its vertex as co-ordinate axes.

If the tangent and normal at a point P of a parabola meet the tangent at the vertex (A) in K and L respectively, prove that

$$K\,L^2 : S\,P^2 :: S\,P - A\,S : A\,S,$$

S being the focus.

16. Find the equations to the tangent and normal at any point of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

If ϕ be the eccentric angle of the point, prove that the equation to the normal is

$$\frac{a\,X}{\cos.\,\phi} - \frac{b\,Y}{\sin.\,\phi} = a^2 - b^2;$$

and hence find the greatest area cut off by the normal from the axes.

1864. *July* 19*th*, *Morn.* 10 *to* 1.—Examiners,—W. H. BESANT, Esq., M.A. and ISAAC TODHUNTER, Esq., M.A., F.R.S.

1. Distinguish between vulgar and decimal fractions, and define a recurring decimal.

Express as a decimal the product of $\frac{3}{5}$, $\frac{9}{8}$, and $\frac{14}{25}$; and divide ·137052 by ·0000324.

Also find the square root of $\cdot\dot{4}$, and of 10·4976.

2. If the 3 per cent. consols be at $91\frac{1}{8}$, what sum of money must be expended in the purchase of stock in order to obtain an income of £528 a year?

If the purchaser afterwards sell out at $92\frac{1}{4}$, and invest the proceeds in mortgages at 5 per cent. per annum, what will be the increase of his income?

3. Prove that $b\,c\,(c-b) + c\,a\,(a-c) + a\,b\,(b-a) = (b-c)(c-a)(a-b)$; and divide $x^6 + 4\,x^5 - 3\,x^4 - 16\,x^3 + 2\,x^2 + x + 3$ by $x^3 + 4\,x^2 + 2\,x + 1$.

Also simplify the expression

$$\frac{x^6 + y^6}{x^6 - y^6} \times \frac{x - y}{x + y} \div \frac{x^4 - x^2\,y^2 + y^4}{x^4 + x^2\,y^2 + y^4},$$

and determine its value when $x = y$.

4. Give the algebraic definition of proportion; and deduce from it Euclid's test of proportion (Def. 5, Book V.).

If $a : b :: c : d$, prove that

$$ma + nb : ma - nb :: mc + nd : mc - nd;$$

and $a^4 + 4a^3b + b^4 : a^4 - 4a^3b + b^4 :: c^4 + 4c^3d + d^4 : c^4 - 4c^3d + d^4$.

5. Find the sum of n terms of a given geometric progression; and, if the common ratio be less than unity, find the sum of the series continued to infinity.

$$\text{Example.} \quad \frac{4}{15} + \frac{1}{5} + \frac{3}{20} + \ldots\ldots \text{ to infinity.}$$

If r be the common ratio, s the sum of n terms, and σ the sum of the squares of the same n terms, prove that

$$\sigma (1 + r)(1 - r^n) = s^2 (1 - r)(1 + r^n).$$

6. Find the number of permutations of n things taken r together.

Four letters are written, and four envelopes directed. Determine (1) the total number of ways in which the letters may be put into the envelopes; (2) the number of ways in which the letters may all go wrong.

7. Define a logarithm; and find the logarithm of 81, (1) to the base 3; (2) to the base $\frac{1}{\sqrt{3}}$.

Prove that

$$\log._a \frac{M}{N} = \log._a M - \log._a N;$$

and having given $\log._{10} 2 = \cdot 301030$, find $\log._{10} 25$, and $\log._{100} 25$.

8. A sum of £P is put out at simple interest for n years. Find an expression for its amount at the end of that time.

If £P be due n years hence, find its present value, reckoning simple interest: and, if the interest be $3\frac{1}{2}$ per cent. per annum, find what must be the value of n in order that the present value may be £$\frac{1}{2}P$.

9. Solve the equations

(α) $(x - a)(x - 2a) = (x - 3a)(x - 4a)$;

(β) $x^2 - 32x + 255 = 0$;

(γ) $x^2 + 2ax + 4a\sqrt{x^2 + 2ax} = 12a^2$;

(δ) $\left\{\begin{array}{r} x^2 + y^2 = 514 \\ xy = 255 \end{array}\right\}$.

10. A certain number, consisting of three digits, exceeds ten times the sum of its digits by 36. The third digit is equal to the sum of the first two; and the second digit increased by unity is equal to the product of the first and third digits. Find the number.

(*Afternoon.—Trigonometry and Conics.*)

11. Define the cosine of an angle. Show that

$$\cos. (A - B) = \cos. A \cos. B + \sin. A \sin. B.$$

Find the cosine of $15°$.

12. Show that in any triangle the sides are proportional to the sines of the opposite angles. Hence, deduce the expression for the cosine of an angle of a triangle in terms of the sides.

13. Find the equations to the straight lines which pass through a given point, and make a given angle with a given straight line.

Example. Find the equations to the lines which pass through the origin, and are inclined at an angle of $75°$ to the straight line $x + y + \sqrt{3}\,(y - x) = a$.

14. Investigate the equation to the tangent at any point of a parabola.

From an external point (h, k) two tangents are drawn to the parabola $y^2 = 4\,a\,x$. Find the area of the triangle formed by the tangents and the chord of contact.

15. Find the equation to the normal at any point of the ellipse.

Write down the equations to the normals at the ends of the latera recta.

16. Find the equation to the hyperbola, considering it as the locus of a point which moves, so that its distance from one fixed point differs by a constant quantity from its distance from another fixed point.

Show that the hyperbola has two asymptotes.

1865. *Tuesday, July* 18*th*.—Examiners,—E. J. Routh, Esq., M.A., and Isaac Todhunter, Esq., M.A.

1. What is the annual interest obtained if £770 14*s*. 7*d*. be invested in the purchase of 3 per cent. stock at $94\frac{5}{8}$?

2. Add together $\frac{5}{16}$, $\frac{7}{32}$, $\frac{3}{8}$, $\cdot046875$, and $1\cdot23$.

Simplify $\dfrac{\cdot0075 \times 2\cdot1}{\cdot0175}$ and $\dfrac{4\cdot255 \times \cdot0064}{\cdot00032}$.

3. Divide $(a^3 - 9\,a^2\,b + 23\,a\,b^2 - 15\,b^3)\,(a - 7\,b)$ by $a^2 - 8\,a\,b + 7\,b^2$.

Find the highest common divisor of $12\,x + 13\,x^2 + 6\,x + 1$ and $16\,x^3 + 16\,x^2 + 7\,x + 1$.

4. Give the algebraical definition of proportion.

If $a : b :: c : d$ and $p : q :: r : s$, show that $\frac{a+c}{b+d} \cdot \frac{p-q}{r-s} = \frac{a-c}{b-d} \cdot \frac{p+q}{r+s}$.

5. Find the sum of a given number of terms of an arithmetical progression, the first term and the common difference being supposed known.

How many terms must be taken of the series 15, 12, 9 . . . that the sum may be 45?

Find the sum of 8 terms of the geometrical progression 4, 2, 1, $\frac{1}{2}$ . . . .

6. Find the number of combinations of n things taken r at a time.

Out of 12 consonants and 4 vowels how many words can be formed, each containing 3 consonants and 2 vowels?

7. Find the present value of an annuity to continue for a certain number of years, allowing compound interest.

If 20 years' purchase must be paid for an annuity to continue a certain number of years, and 24 years' purchase for an annuity to continue twice as long, find the rate per cent.

8. Define a logarithm; and show that in the common system of logarithms the characteristic of a logarithm can be determined by inspection.

Given $\log_{10} 2 = \cdot 301030$, find $\log_{10} \sqrt{\cdot 00025}$.

9. Solve the following equations :—

(1) $(x-10)^3 (x-6) = (x-8)^3 (x-12)$;

(2) $\begin{cases} 3x^2 + 5x - 8y = 36, \\ 2x^2 - 3x - 4y = 3; \end{cases}$

(3) $\frac{a+b}{x+b} - \frac{a+c}{x+c} = \frac{4a}{x+a}$.

10. There is a certain rectangular floor, such that if it had been 2 feet longer and 1 foot narrower the area would have been the same; and if it had been 4 feet shorter and 3 feet broader, the area would also have been the same. Determine the length and breadth of the floor.

(Afternoon.—Trigonometry and Conics.)

11. Define the sine of an angle. Find all the angles whose sine is the same as sine a. Given the tangent of an angle, deduce an expression for the sine of the same angle.

Prove in a geometrical manner that

$$\sin.(A - B) = \sin. A \cos. B - \cos. A \sin. B;$$

where A has a value about 300 and B about 40 degrees.

12. Show how to find the angles of a triangle whose sides are given. Find also an expression for its area.

The diagonals of a quadrilateral field are a and b, and the acute angle between them is θ. Find the area of the field.

13. Find the equation to a straight line in the form $\frac{x}{a} + \frac{y}{b} = 1$.

Give a diagram showing the position of the lines $2x + 3y + 6 = 0$, and $x^2 - y^2 = 0$.

14. Find the equation to the tangent to the parabola $y^2 = 4ax$.

A straight line, $y = bx + c$, is drawn cutting the curve. Prove that the equation to the two straight lines drawn from the origin to the points of intersection are given by

$$c y^2 - 4axy + 4abx^2 = 0.$$

15. Assuming that the sum of the distances of any point of an ellipse from two fixed points, called the foci, is constant, find the equation to the curve in rectangular co-ordinates.

16. The equation to an hyperbola being $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, find the condition that its asymptotes may be at right angles.

1866. *July* 17*th.*—Examiners,—E. J. ROUTH, Esq., M.A., and ISAAC TODHUNTER, Esq., M.A., F.R.S.

1. Simplify

$$\frac{1 + \frac{2}{3} - \frac{3}{4}}{1 + \frac{3}{2} - \frac{4}{3}} \times \frac{1}{1 + \frac{1}{2 + \frac{1}{3}}} \times \left(1 + \frac{1}{1 + \frac{1}{10}}\right).$$

Divide ·00049 by ·07; and find to three places of decimals the value of $(5 + \sqrt{6})(\sqrt{3} - \sqrt{2})$.

2. Reduce 9*s.* 11¾*d.* to the fraction of half a sovereign.

The difference between the simple and compound interest of a certain sum of money for 3 years at 5 per cent. is £1. Find the sum.

3. Simplify

$$\left\{(a+b)^{\frac{1}{3}} - (a-b)^{\frac{1}{3}}\right\}\left\{(a+b)^{\frac{2}{3}} + (a-b)^{\frac{2}{3}} + (a^2 - b^2)^{\frac{1}{3}}\right\};$$

and $(aq - bp)^2 + (br - cq)^2 + (cp - ar)^2 + (ap + bq + cr)^2$, where $a^2 + b^2 + c^2 = 1$, and $p^2 + q^2 + r^2 = 1$.

Reduce to its lowest terms

$$\frac{2x^4 + x^3 + 4x^2 - 1}{x^4 - 2x^3 + 2x^2 - 5x + 2}.$$

4. If $2x + 3y : 3y + 4z : 4z + 5x :: 4a - 5b : 3b - a : 2b - 3a$, prove that $7x + 5y + 8z = 0$.

5. Find the sum of an arithmetical progression, having given the first term, the last term, and the number of terms.

Find the number of terms in an arithmetical series whose first term is -24, common difference 2, and sum -150.

6. Find the number of permutations of n things taken r together.

The contents of a basket containing 12 pears is to be distributed between 12 persons, so that each person is to have one. In how many ways can this be done? If the largest pear be always given to one particular person, in how many ways could the distribution be then effected?

7. Find the present value of an annuity of £100 a year for 10 years, beginning 2 years hence, at 5 per cent. compound interest.

8. Define the characteristic of a logarithm; and find the characteristic of log. ·0003, the base being 10.

Show that log. $M N =$ log. $M +$ log. N.

If log. 2 = ·3010300, and log. 3 = ·4771213, find log. 7·2.

9. Show that for all real values of x the expression $a x^2 + b x + c$ has the same sign as a, if $4 a c - b^2$ be a positive quantity.

Solve the equations

$$x - 3 = 2\sqrt{x};$$

$$x + \sqrt{x^2 - a^2} = \frac{a^2}{2(x + a)};$$

$$\left.\begin{aligned} x^3 + y^3 &= 28 \\ x + y &= 4 \end{aligned}\right\}.$$

10. A person about to invest in the 3 per cent. consols observed that if the price had been £5 less he would have received $\frac{1}{4}$ per cent. more interest for his money. What was the price of consols?

(Afternoon.—Trigonometry and Conics.)

11. Define the sine of an angle; and show, geometrically, that sin. $2A$ is less than 2 sin. A.

Find sin. A from the equation

$$\tan. A + \sec. A = a.$$

12. Show that in any triangle the sides are proportional to the sines of the opposite angles.

One angle of a triangle is 120°, and the sides which contain it are in the ratio of 4 to 1. Show that the cotangents of the other angles are $3\sqrt{3}$ and $\frac{\sqrt{3}}{2}$.

13. Show how to solve a right-angled triangle, having given, a side and an acute angle.

A staff at the top of a tower is observed to subtend an angle of 15° by an observer at a distance of a feet from the foot of the tower, and also to subtend the same angle when the observer is at a distance of b feet. Find the height of the staff.

14. Prove the formula

$$\sin. (A + B) = \sin. A \,.\, \cos. B + \cos. A \sin. B,$$

taking the case in which A and B are each less than a right angle, and their sum greater than a right angle.

If A, B, C are the angles of an acute-angled triangle, show that $\sin. \frac{A}{2} + \sin. \frac{B}{2}$ is greater than $\sin. \frac{C}{2}$.

15. Find the angle between two straight lines expressed by given equations.

Find the equations to the two straight lines which pass through the point (h, k), and make an angle whose tangent is m, with the straight line $y = m\,x + c$.

16. Show that the equation to the circle referred to an origin on the circle is of the form

$$x^2 + y^2 = a\,x + b\,y.$$

If the straight line $y = m\,x + c$ cut the circle, determine the equation which represents the two straight lines from the origin to the points of intersection.

17. Define an ellipse; and obtain its equation in the form

$$a^2\,y^2 + b^2\,x^2 = a^2\,b^2.$$

Show that the equation to the straight line which passes through two points (x_1, y_1) and (x_2, y_2) on the ellipse may be put in the form

$$a^2\,(y - y_1)\,(y + y_2) + b^2\,(x - x_1)\,(x + x_2) = a^2\,y^2 + b^2\,x^2 - a^2\,b^2.$$

Deduce the equation to the tangent at a given point.

1867. *July 16th.—Arithmetic and Algebra.* Examiners,—
E. J. ROUTH, Esq., M.A., and ISAAC TODHUNTER, Esq., M.A., F.R.S.

1. Simplify $\frac{1\cdot18}{\cdot152} \times \frac{3\cdot04}{2\cdot95}$, and divide the result by $\cdot00125$.

2. Find the income arising from investing £740 in the 3 per cents. when they are at $92\frac{1}{2}$.

3. Extract the square root of $4738\cdot02\dot{7}$; and of $4x^4 - 12x^3y + 25x^2y^2 - 24xy^3 + 16y^4$.

4. Simplify $\frac{x^2 - x + 1}{x^2 + x + 1} + \frac{2x(x-1)^2}{x^4 + x^2 + 1} + \frac{2x^2(x^2-1)^2}{x^8 + x^4 + 1}$.

5. If four numbers are proportionals, the product of the extremes is equal to the product of the means.

If a is to b as c is to d, find the relation between p, q, r, and s, in order that
$(pa + qb + rc + sd)(pa - qb - rc + sd)$ may be equal to
$$(pa - qb + rc - sd)(pa + qb - rc - sd).$$

6. Find the number of combinations of n things taken r at a time.

Find how many words of 3 letters each can be formed of 20 consonants and 5 vowels, the vowel being supposed to be always the middle letter of the word.

7. Find the sum of n terms of a geometrical progression, having given the first term and the common ratio.

Find also the sum of the products of every pair of different terms.

8. Find the amount of an annuity left unpaid for any number of years, allowing compound interest.

A person starts with a certain capital, which produces him 4 per cent. per annum compound interest. He spends every year a sum equal to twice the original interest on his capital. Find in how many years he will be ruined, having given log. 2 = $\cdot3010300$, log. 13 = $1\cdot1139434$.

9. Solve the following equations:—

$$(1)\ \frac{1}{x-1} + \frac{2}{x-2} = \frac{3}{x-3};$$

$$(2)\ \frac{9x+20}{36} = \frac{4x+20}{35x+2} + \frac{x}{3};$$

$$(3)\ \frac{1}{x} + \frac{1}{y} = \frac{x+y}{12} = \frac{7}{x+y+5}.$$

10. A horse is sold for £24; and the number expressing the profit per cent. expresses also the cost price of the horse. Find the cost price.

(*Afternoon.—Trigonometry and Conics.*)

11. Define the sine and cosine of an angle. Show that whatever be the magnitude of the angle A, sin. A = cos. $(90° - A)$.

Solve the equations :—

$$\sin.^2 \theta + \cos.^2 (90° - \theta) = 1;$$
$$\tan. \theta = 2 \sin. \theta.$$

12. Show that in any triangle, $\cos. A = \frac{b^2 + c^2 - a^2}{2\,b\,c}$.

If the sides of a triangle were given, would this be a convenient formula to find the angle A? and if in any case it is not, what formula should be used?

The sides of a triangle are 6, 8, 10. Find the greatest angle.

13. Show how to solve a triangle when two sides and an angle opposite to one are given.

Explain how you would find the distance between two inaccessible objects on a level plain.

14. Prove that $\tan. (A - B) = \frac{\tan. A - \tan. B}{1 + \tan. A \tan. B}$.

If $A + B + C = 180°$, prove that

$$\sin.^2 A = \sin.^2 B + \sin.^2 C - 2 \sin. B . \sin. C . \cos. A.$$

15. Find the equation to the straight line the distance of which from the origin is p, and which makes an angle a with the axis of x.

Find the equation to the straight line which joins the intersection of

$$\left.\begin{aligned} 2x + 3\,y - 4 &= 0 \\ x + 2\,y - 1 &= 0 \end{aligned}\right\}$$

to the point $x = 2$, $y = 3$; and give a diagram showing the positions of these three straight lines.

16. Find the general equation to a circle; and investigate the condition that the straight line $y = m\,x + c$ should be a tangent.

17. Find the equation to the parabola in the form

$$y^2 = 4\,m\,x.$$

A normal is drawn at the point whose ordinate is y. Find the coordinates of the other point in which it cuts the curve.

1868. *July* 21*st.*—Examiners,—E. J. ROUTH, Esq., MA., and ISAAC TODHUNTER, Esq., M.A., F.R.S.

1. Find the value of

$$\frac{\cdot 13 \times \cdot 14 \times \cdot 01 - \cdot 12 \times \cdot 14 \times \cdot 02 + \cdot 12 \times \cdot 13 \times \cdot 01}{.01 \times \cdot 2 \times \cdot 01}.$$

Extract the square root of 491401; and reduce 2 days 9 hours to the decimal of a week.

2. A person invested in the 3 per cents. at $94\frac{1}{2}$, and received as interest just £200 a year. What sum did he invest?

3. Simplify $\dfrac{x^3 - 4x^2 + 5x - 2}{x^2 - 1}$; and find the factors of

$$x^3 + 3axy + y^3 - a^3.$$

If $x = \sqrt[3]{-1 + \sqrt{2}} + \sqrt[3]{-1 - \sqrt{2}}$, find the value of $x^3 + 3x + 2$.

4. If $a : b :: c : d$, prove that $a + b : b :: c + d : d$.

What quantity must be added to each of the terms of the ratio $\dfrac{a}{b}$, that it may become the ratio $\dfrac{e}{f}$?

5. Find the sum of a geometrical progression, having given the first and last terms, and the number of terms.

The sum of 40 terms of an arithmetical series is a, and the sum of 50 terms is b. Find the common difference.

6. Find the number of combinations of n things taken r together.

How many different arrangements can be made of the letters of the alphabet, taking them three at a time, two consonants and one vowel being in each arrangement?

7. Investigate the rule to find the discount on any sum £A due t years hence at r per cent. per annum.

8. If log. 4 = ·6020600, log. 27 = 1·4313638, and log. 7 = ·8450980, find log. ·0027 and log. 3528.

Explain what is meant by the modulus of a system of logarithms.

9. Solve the equations:—

$$(1)\quad \frac{x-6}{10} + \frac{x+3}{5} = x - \frac{7}{10};$$

$$(2)\quad \frac{x + \frac{1}{x} - 1}{x - \frac{1}{x} + 1} = 1 - \left(x - \frac{1}{x}\right);$$

(3) $\left\{\begin{array}{l} x^2 + xy = a \\ y^2 + xy = b \end{array}\right\}.$

10. A certain number consists of two digits, and another number is formed from it by reading it backwards. If the sum of the two numbers is 99, and the difference 45, find the digits.

(*Afternoon.—Trigonometry and Conics.*)

11. Determine the values of the trigonometrical ratios for an angle of 60°.

Find A, B, and C, from the equations

$$\cos. (A + B - C) = \tfrac{1}{2}; \ \cos. (A - B + C) = \frac{\sqrt{3}}{2};$$

$$\cos. (A + B) = \sin. C.$$

12. Show how to find the height and the distance of an inaccessible object on a horizontal plane.

A person standing on the bank of a river observes the angular elevation of the top of a tree on the opposite bank to be 60°; and when he retires 100 feet from the edge of the river, he observes the angle to be 30°. Find the height of the tree and the breadth of the river.

13. In any triangle $A\,B\,C$ the tangent of half the difference of the angles B and C is to the tangent of half their sum as the difference of the two sides $A\,B$ and $A\,C$ is to their sum. If $b = 17$, $c = 7$, $a = 60°$, find B and C, having given

$$\log. 2 = \cdot 3010300; \ L \tan. 35° \, 49' = 9\cdot 8583357;$$
$$\log. 3 = \cdot 4771213; \ L \tan. 35° \, 49' \, 10'' = 9\cdot 8583800.$$

14. Find an expression for the area of a triangle in terms of the sides.

The sides of a triangle are in arithmetical progression, and its area is $\frac{3}{5}$ths of that of an equilateral triangle of the same perimeter. Show that the sides of the triangle are as the numbers 7, 10, 13.

15. Investigate the equation to a straight line in the form $\frac{x}{a} + \frac{y}{b} = 1$.

Determine the equation to the straight line which is perpendicular to this straight line, and passes through the point $x = a$, $y = b$.

16. Show that an equation of the form $x^2 + y^2 + A\,x + B\,y = C$ represents a circle. Investigate the locus of a point which moves so that its distance from one fixed point bears a constant ratio to its distance from another.

17. Trace the curve represented by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Find the equation to a straight line which touches this curve and is parallel to the straight line $\frac{x}{a} + \frac{y}{b} = 1$.

1869. *July* 20*th*.—Examiners,—E. J. ROUTH, Esq., M.A., and Prof. H. J. S. SMITH, M.A., F.R.S.

1. Prove the rule for dividing one fraction by another; and find the value of

$$\frac{\cdot 05 \times \cdot 05 \times \cdot 05 + 1}{1\cdot 05},$$

and of $\cdot\dot{4}2857\dot{1}$ of 1 minute 17 seconds.

2. Divide £26 3*s.* 3*d.* between 3 persons, so that their shares may be to one another in the proportion of the fractions $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$. Extract the square root of 17 as far as four places of decimals.

3. Simplify the expressions:—

(1) $$\frac{a^2}{(a-b)(a-c)} + \frac{b^2}{(b-c)(b-a)} + \frac{c^2}{(c-a)(c-b)},$$

(2) $$\frac{\sqrt{12}}{(1+\sqrt{2})(\sqrt{6}-\sqrt{3})}.$$

If $x^2 + 2ax - 3b^2$ is divisible by $x - a$ without remainder, show that a is equal either to $+b$ or to $-b$.

4. If $a : b :: b : c :: c : d$, prove that $a : d :: a^3 : b^3$. A varies partly directly as B, and partly inversely as B. If A is 3 when B is 1, and also when B is 2, what will be the value of B when A is $4\frac{1}{2}$?

5. Prove the rule for finding the sum of n terms of an arithmetical series.

The first term of a geometric series is 3, and its fourth term is $\frac{1}{\sqrt{3}}$: find its sum to infinity.

6. Show that the number of combinations of n things taken r together is the same as the number of combinations of n things taken $n - r$ together.

Given 10 white and 10 black balls: in how many different ways can I select from them a set of 10 balls, of which 5 shall be white and 5 black?

7. Find the present value of an annuity of £A, to continue for n years, allowing compound interest at the rate of r per cent. per annum.

8. Solve the equations:—

$$(1)\quad \left.\begin{aligned} \frac{x}{a} - \frac{y}{b} &= 1 + \frac{a^2}{b^2} \\ \frac{x}{b} + \frac{y}{a} &= 1 + \frac{b^2}{a^2} \end{aligned}\right\};$$

$$(2)\quad \frac{x^2 + x + \frac{1}{2}}{a^2 + 1} + \frac{x^2 + x}{a^2 - 1} = 0;$$

$$(3)\quad x + y = \frac{1}{x} + \frac{1}{y} = \frac{5}{2}.$$

9. A rectangular court is 10 yards longer than it is broad; its area is 1,131 square yards. What is its length and breadth?

10. Define a logarithm; and prove that the logarithm of a product of 2 factors is equal to the sum of the logarithms of the 2 factors.

Given $\log._{10} 2 = \cdot 30103$, what are the logarithms of $2 - \frac{1}{5}$, of $\cdot 00002$ of $62\cdot 5$ and of $5 -$?

(Afternoon.—Trigonometry and Conics.)

11. The sine of an unknown angle x being given equal to sin. a where a is given, investigate a general expression for the angle x.

Solve the equations:—

$$\sin. x + \cos. x = 1;$$
$$\cos. 2\,x = \cos.^2 x.$$

12. Show that $\cos. A + \cos. B = 2 \cos. \dfrac{A+B}{2} \cos. \dfrac{A-B}{2}$;

and investigate a corresponding expression for cos. A — cos. B.

Show also that

$$\sin. (A + B) \sin. (A - B) = \sin.^2 A - \sin.^2 B;$$
$$\sqrt{1 - \sin. 2\,A} = \cos. A - \sin. A.$$

13. Three inaccessible objects, A, B, C, are on a level plain, and their distances are known by means of a map.

The angles $A\,O\,B$, $B\,O\,C$, being observed at some place O, show how to find the distances $A\,O$, $B\,O$, $C\,O$, by formulæ adapted to logarithmic calculation.

14. Given in a triangle two sides and the included angle, investigate a formula to find the difference of the other two angles of the triangle.

In any triangle show that

$$\tfrac{1}{2}(a^2 + b^2 + c^2) = b\,c \cos. A + c\,a \cos. B + b\,a \cos. C.$$

15. Find the equation to the straight line which passes through the point whose coordinates are a and b, and is parallel to $A\,x + B\,y + C = 0$.

16. Give a diagram showing the position of the circle

$$x^2 + y^2 - 2\,x - 2\,y + 1 = 0;$$

and determine whether the straight line

$$x + y = 2 + \sqrt{2}$$

is a tangent or not.

17. Find the locus of a point, P, which moves so that the sum of its distances from two fixed points, A and B, is constant.

1870. *July* 19*th*.—Examiners,—Prof. H. J. S. SMITH, M.A., F.R.S., and Prof. SYLVESTER, M.A., F.R.S.

1. Prove the rule for finding the greatest common measure of two numbers.

Reduce the fraction $\dfrac{17427}{24975}$ to its lowest terms.

2. ( ) If 640 acres go to a square mile, what is the length of the sides of a square plot of ground which contains 100 acres?

(β) Find the square root of the circulating decimal 111 . . . . In applying the ordinary method of extracting the square root to this example, state any law that you notice in the form of the digits which express the successive remainders.

3. Simplify the expressions

$$(\alpha)\quad \frac{a+b}{a-b} + \frac{b+c}{b-c} + \frac{c+a}{c-a} + \frac{(a+b)\,(b+c)\,(c+a)}{(a-b)\,(b-c)\,(c-a)}.$$

$$(\beta)\quad \left(\frac{a}{b}+\frac{b}{a}\right)^2 + \left(\frac{b}{c}+\frac{c}{b}\right)^2 + \left(\frac{c}{a}+\frac{a}{c}\right)^2 - \left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right).$$

4. (α) If $A \propto B$ when C is invariable, and $A \propto C$ when B is invariable, prove that $A \propto B\,C$ when B and C are both variable.

(β) The total increase in the number of patients in a certain hospital this year over the number in the year preceding was $2\frac{1}{2}$ per cent.; in the number of out-patients there was an increase of 4 per cent.; but in that of the in-patients a decrease of 11 per cent. Find the ratio of the number of out-door to the number of in-door patients.

5. (a) In former times troops, for the purpose of making a stand on all sides, used to be drawn up in the form of a solid triangle, 1 man in the first rank, 3 men in the second rank, 5 men in the third rank, and so on. Prove that a triangular battalion so formed would always admit of being transformed into a solid square.

(β) Prove that if the squares of three quantities be in arithmetical progression, so also will be the reciprocals of their sums taken, two and two together; and give a numerical illustration.

6. A committee of 7 members is to be chosen out of a body composed of 20 Protestants and 15 Catholics, in such a way that there shall be 3 of one creed, and 4 of the other, on the committee. In how many different ways can such a committee be constituted?

7. (a) Find the value of a perpetual annuity of £225 per annum, at $3\frac{1}{4}$ per cent. rate of interest.

8. Solve the equations :—

(a) $$\frac{x}{a} + \frac{y}{b} = 1 - \frac{x}{c} = 1 + \frac{y}{c};$$

(β) $$\frac{1}{\sqrt{x} - \sqrt{2 - x}} - \frac{1}{\sqrt{x} + \sqrt{2 - x}} = 1.$$

(γ) $$\left.\begin{array}{r} xy = 12 \\ x^5 - y^5 = 781 \end{array}\right\}.$$

9. (a) An express train which ought to travel at uniform speed, after being an hour in motion, was delayed half-an-hour by an accident; after which it proceeded at three-fourths of its original rate of speed, and, in consequence, arrived at the end of its journey 1 hour 50 minutes behind time. Had the accident occurred (and the same delay and subsequent retardation taken place) after the train had travelled a distance of 60 miles, it would have been 1 hour 40 minutes behind time. Find the length of the line.

(β) Supposing the above question were varied in the latter part of it by your being informed that "had the accident occurred when the train had gone *half way*, it would have arrived 1 hour 20 minutes behind time;" would that information have been incorrect? Would it have enabled you to determine the length of the line?

10. What is the characteristic of the logarithm of 2000 to the base 3? If the mantissæ of logarithms of 9450, 9451 to the base 10 are 9754318, 9754778 respectively, find the complete logarithm to the same base of 9450666 by the method of proportional parts.

(Afternoon.—Trigonometry and Conics.)

11. Given the co-ordinates of two points A and B, obtain the co-ordinates of the point which divides the straight line $A\,B$ in a given ratio.

One vertex of a parallelogram is at the origin; the co-ordinates of the two vertices adjacent to this vertex are respectively (x_1, y_1) (x_2, y_2). Find the co-ordinates of the remaining vertex.

12. Find the co-ordinates of the centre, and the radius, of the circle, $x^2 + y^2 + 2\,x - 6\,y = 0$. Trace this circle, and state in what point it cuts the axes. Prove that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the point whose co-ordinates are $(a \cos. a, b \sin. a)$; and find the equation of the line which touches the ellipse at this point.

13. Define the cosine of an angle; and trace the variations in sign and magnitude of the cosine, as the angle increases from 0° to 180°.

14. Show that in any triangle, of which the sides are a, b, c, and the angles opposite to them A, B, C,

$$(a) \qquad \frac{\sin. A}{a} = \frac{\sin. B}{b} = \frac{\sin. C}{c};$$

$$(\beta) \qquad \tan. \left(\frac{A - B}{2}\right) = \frac{a - b}{a + b} \cot. \frac{C}{2}.$$

15. A tower 100 feet high is observed at a station A, which is on a level with the base of the tower; and the angle of elevation of the top of the tower is found to be 45°. The observer then proceeds from A to B in a direction at right angles to the line joining A to the base of the tower; and he finds that at the station B (which is on the same level as A) the angle of elevation of the top of the tower is 30°. What is (approximately) the distance between the stations A and B?

1871. *July* 18*th*.—Examiners,—Prof. H. J. S. SMITH, M.A., F.R.S., and Prof. SYLVESTER, LL.D., F.R.S.

1. Find the greatest common divisor of 1,287,000, and 504,504; and prove that every common divisor of two given numbers divides their greatest common divisor.

2. (α) How long will an up train and a down train be in passing one another, if each of them be 44 yards long, and if each of them travels at the rate of 30 miles an hour?

(β) A metre being 39·370 inches, state accurately, as far as three places of decimals, what decimal fraction a foot is of a metre?

3. The population of a country is at present 32,000,000, and increases at the rate of 5 per cent. every year, what will it be at the end of 5 years?

4. Simplify $$\frac{(ay - bx)^2 + (ax + by)^2}{\left(\frac{a}{b} + \frac{b}{a}\right)\left(\frac{x}{y} + \frac{y}{x}\right)};$$
and divide $1 + 10x^3 + 27x^6$ by $1 - 2x + 3x^2$.

5. If 4 numbers are proportionals, prove that

(1) Their reciprocals are proportionals;

(2) The greatest and least of them are together greater than the other two.

6. How many different arrangements can there be of n letters, $a, b, c, \ldots\ldots$?

In how many of these arrangements will a and b be next to one another? In how many of them will a come before b (but not necessarily immediately before b)?

7. Sum the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$ to n terms; and show that the sum of any odd number of terms of this series is always greater, the sum of any even number of terms always less, than the sum to infinity.

What is the least number of terms of the series which will give a sum differing from the sum to infinity by less than ·0001?

8. Solve the equations:—

(α) $$\frac{x - 1}{x + 1} + \frac{x + 1}{x - 1} = \frac{5x}{x^2 - 1};$$

(β) $$\left.\begin{aligned} \frac{x}{a + 2} + \frac{y}{a} &= 1 \\ \frac{x}{a} + \frac{y}{a - 1} &= 1 \end{aligned}\right\}.$$

Is there any value of a for which the equations (β) are not resoluble?

9. An annuity of £P per annum is to begin n years hence, and is to be payable for ever. Find its present value at r per cent. rate of interest; and show that its present value is to its value n years hence as $(1 + r)^{-n} : 1$.

10. What is meant by the base of a system of logarithms? What are the advantages of taking 10 as the base?

Prove that if $N = \frac{P}{Q}$, log. N = log. P — log. Q; and find as far as four places of decimals, the number of which the logarithm to base 10 is ·5.

(Afternoon.—Trigonometry and Conics.)

11. Find the perpendicular distance of the point ξ, η from the line $y = a\,x + b$.

Determine the locus of a point equi-distant from a point and a right line.

12. Prove analytically that the three lines drawn from the angles of a triangle, whether to the middle points of the opposite sides, or perpendicular to those sides, meet in a point.

13. Obtain the equation to the line joining the centres of the two circles.

$$x^2 + y^2 + 2\,a\,x + 2\,b\,y + c = 0,$$
$$x^2 + y^2 - 2\,b\,x - 2\,a\,y + c = 0;$$

and find the relation between a, b, c, when these two circles touch each other.

14. Obtain the equation to a tangent equally inclined to the major and minor axis of the ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

How many such tangents can be drawn, and what is the area of the figure which they inclose?

15. Explain how, in analysis, an angle is regarded as susceptible of assuming all degrees of magnitude between positive and negative infinity.

What are the limits to the magnitude of an angle in Euclidean Geometry?

From the general expression for the sine of the sum of two angles, deduce the formula for the cosine and tangent of their difference.

16. Two sides and an included angle of a triangle being given, show how to solve the triangle.

What is meant by the "Ambiguous" case in the solution of triangles? What parts are given when the case arises? These

parts being given, is the solution necessarily ambiguous? If not, determine the conditions of the ambiguous case arising.

17. Two cliffs stand facing each other on opposite sides of a river. From the top of one of them, known to be 200 feet high, the angles of depression of the summit and foot of the other are observed to be 30° and 45° respectively. Find in feet and inches the height of the latter and the breadth of the river.

1872. *July* 16*th*.—Examiners,—Prof. H. J. S. SMITH, LL.D., F.R.S., and Prof. SYLVESTER, LL.D., F.R.S.

1. State and prove the rule for the division of decimals.

If the length of the year is 365·242264 days, but is reckoned as equal to $365\frac{1}{4}$ days, find in how many centuries the accumulated error would amount to $263\frac{3}{125}$ days.

2. Find the greatest common measures of 11,310 and 86,478, of 86,478 and 448,630, and of 11,310, 86,478, 448,630.

Find the least common multiple of $10\frac{1}{2}$, $6\frac{7}{8}$, $4\frac{9}{10}$.

3. Simplify $\sqrt{\dfrac{147}{605}}\left(\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}-\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}\right)$.

Express the fraction $\dfrac{1-\sqrt{2}+\sqrt{5}}{1+\sqrt{2}-\sqrt{5}}$ under the form of a fraction with a rational denominator.

4. Find the value of

$$\frac{(a^2+b^2-c^2)(a^2+c^2-b^2)+(b^2+c^2-a^2)(b^2+a^2-c^2)+(c^2+a^2-b^2)(c^2+b^2-a^2)}{(a+b+c)(a+c-b)(b+c-a)(a+b-c)};$$

and divide $\dfrac{x-a}{x+a}-\dfrac{x^3-a^3}{x^3+a^3}$ by $\dfrac{x+a}{x-a}+\dfrac{x^2+a^2}{x^2-a^2}$.

5. If 4 quantities are proportionals, and the second of them is a mean proportional between the third and fourth, prove that the third will be a mean proportional between the first and the second.

6. Find the number of permutations, and also the number of combinations, of n things, taken m at a time.

With 17 consonants and 5 vowels, how many words can be formed having 2 different vowels in the middle, and 1 consonant (repeated or different) at each end?

7. Find the sum of a geometrical progression to n terms.

What is the value of the series $\frac{1}{2}-\frac{1}{3}+\frac{2}{9}-\frac{4}{27}\ldots\ldots$ *ad infinitum*?

8. If an annuity continued for ever is worth 25 years' purchase, what annuity (reckoning at the same rate of interest), to continue for 3 years, can be purchased for £5,000?

A man invests £10,000 in land; he borrows $\frac{5}{7}$ths of the value of his new investment; and so on continually. What would be the aggregate amount borrowed if this process were continued indefinitely?

9. Solve the equations:—

$$\frac{x-y}{x\,y} = \tfrac{1}{6},\quad \frac{x-z}{x\,z} = \tfrac{2}{3},\quad \frac{y+z}{y\,z} = 1\tfrac{1}{2}.$$

The sum of 3 numbers in arithmetical progression is 33, and the sum of their squares is 435. Find the common difference.

10. What is the characteristic of the logarithm of 50 to the base $\sqrt{2}$? If the logarithm of 3 to the base 10 is ·477121, what is its logarithm to the base $\sqrt[5]{10}$?

Given log. 648 = 2·81157501,

log. 864 = 2·93651374,

find log. 108.

(*Afternoon.—Trigonometry and Conics.*)

11. The co-ordinates of the points P and Q being (a, b) and (b, a) respectively, and O being the origin, find the equations of the lines $O\,P$, $O\,Q$, $P\,Q$, and the area of the triangle $O\,P\,Q$.

12. If (x, y) are the co-ordinates of a point P upon an ellipse, of which the equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, find at what distances from the origin the axis major is cut by the tangent and by the normal at P; and show that the rectangle contained by these distances is equal to the difference between the squares of the semi-axes of the ellipse.

13. Prove the formula

$$\sin.(A + B) = \sin. A \cos. B + \sin. B \cos. A,$$

drawing the figure for the case in which A and B are each less than 90°, but $A + B$ greater than 90°.

Find the sine and cosine of 75°.

14. Given in a plane triangle, a, B, and A, solve the triangle.

If the sides of a triangle are 9 feet, 7 feet, and 4 feet respectively, what are the sines of the angles of the triangles?

1873. *July* 23*rd*.—Examiners,—Prof. H. J. S. SMITH, LL.D., F.R.S., and Prof. SYLVESTER, LL.D., F.R.S.

1. State and prove the rule for fixing the position of the decimal point in the quotient obtained by dividing one decimal by another.

Find the value of

$$\frac{\cdot011 \times 133\cdot1 - \cdot723 \times \cdot00723}{1\cdot1377};$$

and express, as a fraction of nine seconds, $\cdot00002578125$ of $3\frac{1}{3}$ days.

2. Find the vulgar fraction equivalent to $\cdot0\dot{7}148285\dot{}$, and the circulating decimal equivalent to $\frac{1}{\cdot1001}$.

If n is a whole number, state in what cases the decimal equivalent to $\frac{1}{n}$ terminates, and in what cases it circulates.

Prove also that the decimal equivalent to $\sqrt{2}$ can never either terminate or circulate.

3. Extract the square root of $6\cdot33679929$, and of

$$x^4 + x^3 + \tfrac{5}{4}x^2 + \tfrac{5}{2}x + \tfrac{5}{4} + \frac{1}{x} + \frac{1}{x^2}.$$

4. Simplify

(α) $(x-y)^3 + (y-z)^3 + (z-x)^3 - 3(x-y)(y-z)(z-x)$;

(β) $$x^3 + \cfrac{x^2}{x^2 + \cfrac{1}{x^3 - \cfrac{x^8 + x^3 - 1}{x^5}}};$$

(γ) $$\frac{1}{a-2b} - \frac{2}{a-b} + \frac{2}{a+b} - \frac{1}{a+2b}.$$

5. Solve the equations :—

(1) $$\frac{2x^2 - 3x + 1}{x^2 - 2x + 2} = \frac{2x-3}{x-2};$$

(2) $$2x^2 - 21x + 55 = 0;$$

(3) $$\left.\begin{aligned} 2x + y + z &= a \\ x + 2y + z &= b \\ x + y + 2z &= c \end{aligned}\right\}.$$

6. The arithmetical mean between two numbers is $1 + a^2$, and the geometrical mean is $1 - a^2$. What are the numbers?

Three numbers are in geometric progression: the common ratio is equal to the first, and also to nine-tenths of the sum of the second and third. Find the three numbers.

7. Prove the formula for the number of combinations of n things taken r and r together.

In how many ways can I select two white balls and three red out of an urn containing seven white balls and ten red?

8. When is one quantity said to vary as another?

If A varies as B^2, B^3 as C^4, C^5 as D^6, and D^7 as E^4, show that $\frac{A}{E} \times \frac{B}{E} \times \frac{C}{E} \times \frac{D}{E}$ does not vary at all.

9. What annuity, beginning n years hence and lasting for n years, is equal in value to an annuity of £A beginning now and lasting for n years, interest being reckoned at $R - 1$ per cent?

10. Given log. $2 = \cdot 301030$, log. $3 = \cdot 477121$, find the logarithms of $\cdot 00625$, of $\frac{1}{24}$, and of $(\cdot 0003)^5$. Find also approximately the value of x which satisfies the equation $2^x = 5$.

(Afternoon.—Trigonometry and Conics.)

11. Given the two right lines $a\,x + b\,y = c$, $a\,y - b\,x = c$, determine their mutual inclination and point of intersection. Find also the equation to a line bisecting internally or externally the angle at which they meet.

12. Obtain the general equation to a circle referred to rectangular coordinates.

The equation of a circle being

$$\sqrt{1 + m^2}\,(x^2 + y^2) - 2\,c\,x - 2\,m\,c\,y = 0,$$

find its radius.

13. Define the *focus*, *directrix*, *axis*, and *latus rectum* of a parabola; and obtain its equation referred to its axis and directrix as axes of coordinates. Find also the equation to the line joining the origin of this system of coordinates with an extremity of the *latus rectum*, and show that it will touch the curve.

14. Prove the formula

$$\cos.\ 3\,\theta = 4\,(\cos.\ \theta)^3 - 3\cos.\ \theta,$$

and apply it to find the values of sin. 18°, cos. 36°.

15. The three sides of a triangle being given, obtain formulæ for the sines, cosines, and tangents of the semi-angles. Which of these formulæ is to be preferred in computing the angles by means of logarithms, and why?

1874. *July* 22*nd*.—Examiners,—Prof. Henrici, Ph.D., F.R.S. and Prof. Sylvester, LL.D., F.R.S.

1. State and prove the rule for finding the greatest common measure of two numbers.

2. A person inquiring the time of day is told that it is between 5 and 6, and that the hour and minute hands of the clock are together. What o'clock is it?

3. Simplify $\dfrac{\sqrt{a+b}+\sqrt{a-b}}{\sqrt{a+b}-\sqrt{a-b}}-\dfrac{\sqrt{a+b}-\sqrt{a-b}}{\sqrt{a+b}+\sqrt{a-b}}$.

Divide $a^2-b^2-c^2-2bc$ by $\dfrac{a+b+c}{a+b-c}$. Find the square of

$$\frac{\sqrt{2ab+(a^2+b^2)\sqrt{-1}}+\sqrt{2ab-(a^2-b^2)\sqrt{-1}}}{a+b}.$$

4. If a, b, c, d, e, f are all positive, prove that $\dfrac{a+c+e}{b+d+f}$ lies between the greatest and least of the fractions $\dfrac{a}{b}$, $\dfrac{c}{d}$, $\dfrac{e}{f}$.

5. Find the sum of $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots$ to infinity; and the sum of the least number of terms of the series differing by less than $\frac{1}{1000}$ from the sum to infinity.

6. If $(a+b+c+d)(a-b-c+d)=(a+b-c-d)(a-b+c-d)$, prove that a, b, c, d are proportionals. If x, y, z are variable, but their sum constant, and if $(x-y+z)(x+y-z)$ varies as yz, prove that yz varies as $(y+z-x)$.

7. Find the number of permutations and of combinations of n things taken i and i together. (2) Twelve balls are to be separated into 3 heaps containing 3, 4, 5 balls respectively. In how many ways can this be effected?

8. What is the present value of an annuity of a given amount per annum in perpetuity, when the rate of interest is r per cent.? If such an annuity is worth 25 years' purchase, what is the value of an annuity of £1 at the end of the 1st year, £2 at the end of the 2nd year, £3 at the end of the 3rd year, and so continued for ever?

9. Solve the equations,

$$(\alpha)\quad \sqrt{x^2-8x+31}+(x-4)^2=5;$$

$$(\beta)\quad \frac{x^2}{y}-\frac{y^2}{x}=28,\qquad x-y=8.$$

10. The mantissæ of the common logarithms of 2, 3, 11 appear from the tables to be ·301030, ·477121, ·041393. Find in how many years the population of a country will first become increased by more than one half of its original amount, through the sole effect of births and deaths, when the birth-rate at the end of each year is $\frac{1}{30}$th, and the death-rate $\frac{1}{40}$th, of the population at the beginning of that year.

(*Afternoon.—Trigonometry and Conics.*)

11. Show that the straight line $4x - y = 17$ passes through the centre of the circle $x^2 + y^2 - 8x + 2y = 0$.

Find the equation to the diameter at right angles to that line, and the co-ordinates of the points where it cuts the circle.

12. Prove that the straight line $10x - 3y = 32$ touches the hyperbola $\frac{x^2}{16} - \frac{y^2}{64} = 1$, and find the co-ordinates of the point of contact.

Find also the co-ordinates of the points where it cuts the asymptotes of the hyperbola, and show that these are equidistant from the point of contact.

13. Prove geometrically that

$$\cos. (A - B) = \cos. A . \cos B + \sin. A . \sin. B,$$

supposing A and B to be both angles in the second quadrant.

14. Given the area of a triangle and two of its sides, show how to find the angles and the third side.

1875. *July* 21*st.*—Examiners,—Prof. HENRICI, Ph.D., F.R.S., and Rev. Prof. TOWNSEND, M.A., F.R.S.

1. Find the prime factors of 6930, 1470, and 5775; and use them for calculating (1) the sum of the reciprocals, and (2) the square root of the product of the three numbers.

2. Extract the square root of 10 by the ordinary rules to four places of decimals.

How many more places can you get by simple division? By aid of the value of $\sqrt{10}$ thus obtained find $\sqrt{\cdot004}$.

3. Simplify the expression

$$\left(\frac{y-z}{x}+\frac{z-x}{y}+\frac{x-y}{z}\right)\left(\frac{x}{y-z}+\frac{y}{z-x}+\frac{z}{x-y}\right)+\left(\frac{z-x}{x-y}-\frac{x-y}{z-x}\right)\left(\frac{y}{z}-\frac{z}{y}\right)$$
$$+\left(\frac{x-y}{y-z}-\frac{y-z}{x-y}\right)\left(\frac{z}{x}-\frac{x}{z}\right)+\left(\frac{y-z}{z-x}-\frac{z-x}{y-z}\right)\left(\frac{x}{y}-\frac{y}{x}\right).$$

4. Prove, if the four fractions

$$\frac{bx+cy+dz}{b+c+d-a},\ \frac{cx+dy+az}{c+d+a-b},\ \frac{dx+ay+bz}{d+a+b-c},\ \frac{ax+by+cz}{a+b+c-d}$$

are equal to one another, their common value will be equal to $\frac{x+y+z}{2}$, as long as $a+b+c+d$ does not vanish; but if $a+b+c+d=0$, the quantities x, y, z must be equal to one another; and then half their common value will be the common value of the fractions.

5. With five dice how many different throws are possible? How many throws in which no two dice show the same number of eyes? and distinguishing the different dice, in how many different ways may one of the latter throws be obtained?

6. Having given the first term, the common difference, and the sum of an arithmetical progression, find the number of terms.

If the first term be equal to 27, the fourth term equal to 18, and the sum equal to 117, find the number of terms and the last term.

7. Find the present value of an annuity of £20 for five years at $3\frac{1}{4}$ per cent., to commence at the end of twenty years.

What would be the value if a half-yearly payment of £10 were substituted for a yearly payment of £20?

8. Solve the equations

$$\left.\begin{aligned}\frac{x}{b+c}+\frac{y}{c-a}&=a+b\\ \frac{y}{c+a}+\frac{z}{a-b}&=b+c\\ \frac{z}{a+b}+\frac{x}{b-c}&=c+a\end{aligned}\right\}.$$

Also

$$\frac{1}{x+\cfrac{1}{y-\cfrac{1}{x}}}=\frac{1}{x-\cfrac{1}{y-\cfrac{1}{x}}};\quad \frac{1}{y}\left(1-\frac{1}{x}\right)=1.$$

9. Find the condition that a quadratic equation,

$$a x^2 + 2 b x + c = 0,$$

may have equal roots; and, this condition being satisfied, find the roots. Write down a quadratic equation which has the roots $a - \beta$ and $a + \beta$.

10. Solve the equations

$$\left.\begin{array}{l}\left(\dfrac{24}{x}\right)^2 + (y-4)^2 = 65 \\ \left(\dfrac{12}{x}\right)^2 + 9 = (5y - 20)^2\end{array}\right\}.$$

11. What is meant by the logarithm of a number? and what by a system of logarithms?

Having given a system of logarithms to the base a, how may the logarithms to the base b be calculated?

12. Having given log. $3796 = 3{\cdot}5793262$,
log. $2984 = 3{\cdot}4747988$,
log. $90714 = 4{\cdot}9576743$,
log. $90715 = 4{\cdot}9576791$,

calculate to seven decimals

$$\sqrt[5]{\frac{({\cdot}3796)^3}{({\cdot}2984)^2}}.$$

(*Afternoon.—Trigonometry and Conics.*)

13. A circle, in a plane, has its centre at the point whose rectangular co-ordinates are a and b, and passes through the origin; find its equation, that of the tangent to it at the origin, and the lengths of the intercepts it cuts off on the axes.

14. A parabola, in a plane, touches the two axes of co-ordinates, supposed rectangular, at two points distant by intervals a and b from the origin. Find its equation, that of the right line through the origin parallel to its axis, and the co-ordinates of its focus.

15. An ellipse and an hyperbola, both in the same plane, have their foci at the same two points on the axis of x equidistant in opposite directions by the interval c from the origin, and pass through the point whose rectangular co-ordinates are x' and y'. Find their equations, those of the tangents to them at the point, and the lengths of their semi-axes.

16. Assuming the numerical value of π to five decimal places, calculate, to the nearest integer, the number of seconds in the angle subtended at the centre of a circle by an arc of length equal to that of the radius of the circle.

17. Assuming the fundamental trigonometrical formulæ for sin. $(A+B)$ and for cos. $(A+B)$ in terms of the sines and cosines of A and B, deduce from them those for sin. $3A$ and for cos. $3A$ in terms of sin. A and of cos. A respectively.

12. Given, of a plane triangle, the base c, and the two base angles A and B, express, in terms of them, the two sides a and b, the altitude h, and the area of the triangle.

1876. *Wednesday, July 19th.*—Examiners, Prof. HENRICI, PH.D., F.R.S., and Rev. Prof. TOWNSEND, M.A., F.R.S.

1. Simplify, by reduction, the algebraical sum:

$$\frac{(x^2-yx)}{(x-y)(x-z)}+\frac{(y^2-zx)}{(y-z)(y-x)}+\frac{(z^2-xy)}{(z-x)(z-y)}.$$

2. Verify, by multiplication, the algebraical identity:

$$(bcx+cay+abz+xyz)^2+(ayz+bzx+cxy-abc)^2$$
$$=(a^2+x^2)(b^2+y^2)(c^2+z^2)$$

3. Extract the square root of $1+2x+3x^2+4x^3+$ &c., to infinity (where x is a proper fraction less than unity), by the ordinary process.

4. Reduce to a simple equation and solve for x, from

$$\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-4}{x-5}-\frac{x-5}{x-6}.$$

5. Reduce to a quadratic equation and solve for x, from

$$\frac{x+a}{x-a}+\frac{x+b}{x-b}+\frac{x+c}{x-c}=3.$$

6. The sum of two numbers is 47·437, and their product is 486·7641. Determine them correctly, each to three decimal places.

7. Assuming that one rood of superficial area $=40$ square perches, and that one square perch $=30\frac{1}{4}$ square yards; calculate, to the nearest integer, the length in feet of each side of a square court-yard of one rood in extent.

8. An alphabet being supposed to contain m consonants and n vowels, required the entire number of different words, each containing p of the former and q of the latter, that could be formed out of its letters.

9. The interest (x) on either (a) of two given sums of money, a and b, being supposed equal to the discount (y) on the other (b) for any fractional period (k) of one year; required, in terms of a, b, and k, the value of x or y.

10. State and prove the ordinary formula for the present value, at n per cent. compound interest, of a deferred annuity, to commence at the expiration of p, and to continue for the duration of q, years.

11. If x be a large positive number, and $\pm\delta$ a comparatively small increase or diminution of it; prove the approximate formula, $\log(x\pm\delta)=\log x\pm\mu\frac{\delta}{x}$, where $\mu=$ the modulus of the system employed.

12. Given that, to five decimal places, $\mu=\cdot43429$ in the ordinary system for which the base$=10$; calculate, to the same number of places, the logarithms of 999 and of 1001 in that system.

(Afternoon.—Trigonometry and Conics.)

13. Assuming the fundamental formulæ of trigonometry, prove the formula

$$\cos. A+\cos. B=2\cos.\frac{A+B}{2}\cos.\frac{A-B}{2}.$$

14. If A, B, C are the angles of a triangle, show that

$$\sin. A\sin. B\sin. C=\sin. A\cos. B\cos. C+\sin. B\cos. C\cos. A+\sin. C\cos. A\cos. B.$$

15. By aid of trigonometrical formulæ obtain the expression for the area of a triangle in terms of its sides.

16. Prove the following statements regarding the lines whose equations are

$$15x-18y+1=0,$$
$$12x+10y-3=0,$$
$$6x+66y-11=0;\text{ viz.:}$$

(1) The three lines meet in a point;
(2) The first two are perpendicular to each other;
(3) The third bisects the angle between the other two.

17. Find, by elementary or co-ordinate geometry, the locus of a point whose distances from two fixed points are in a constant ratio.

18. At the points where the line $\frac{x}{a}+\frac{y}{b}=0$ cuts the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ tangents are drawn. Find their equations, and prove that they are parallel.

1877. *July* 18*th*.—Examiners, Prof. HENRICI, PH.D., F.R.S., and Rev. Prof. TOWNSEND, M.A., F.R.S.

1. Find to the nearest shilling the present value of £273, payable after three years, the rate of interest being three per cent. per annum.

2. Extract the square roots of $2+\sqrt{3}$ and $2-\sqrt{3}$.

3. Having given:

$$\log.\ 193{\cdot}06 = 2{\cdot}2856923,$$
$$\log.\ 19{,}307 = 4{\cdot}2857148,$$

find the seventh root of 100 to six decimals.

4. If $\kappa = x\sqrt{1+y^2} + y\sqrt{1+x^2}$, prove that

$$\sqrt{1+\kappa^2} = xy + \sqrt{1+x^2}\,\sqrt{1+y^2}.$$

5. If x, y, z be proportional to given quantities A, B, C, respectively, and if the sum of their squares be $=100$, find their values.

6. Three persons have four coats, five vests, and six hats between them: in how many different ways can they dress themselves with them?

7. Solve the equations:

$$\left.\begin{aligned} 3x+9y-\ 6z &= \ 8, \\ 4x-7y+13z &= \ 9, \\ 12x-3y-\ 5z &= 11. \end{aligned}\right\}$$

8. Find all the solutions of the system of equations:

$$x^2+xy=(a-b)^2, \qquad xy+y^2=4ab.$$

9. Out of a cask containing 360 quarts of pure alcohol a quantity is drawn off and replaced by water. Of the mixture a second quantity, 84 quarts more than the first, is drawn off and replaced by water. The cask now contains as much water as alcohol; find how many quarts were taken out the first time.

Show that the problem has only one solution.

10. Find the L.C.M., and the factors, of the expressions:

$$3x^3+x^2-8x+4, \quad 3x^3+7x^2-4, \quad x^3+2x^2-x-2, \quad 3x^3+2x^2-3x+2;$$

and also the factors of their sum. Find the number of their reciprocals.

11. Convert the circulating decimal fraction $1{\cdot}46\dot{3}\dot{1}$ into a vulgar fraction. Find also the sum of:

$$\frac{1}{8}+\frac{4}{8^2}+\frac{6}{8^3}+\frac{3}{8^4}+\frac{1}{8^5}+\frac{4}{8^6}+\frac{6}{8^7}+\frac{3}{8^8}+\frac{1}{8^9}+\ \ldots\ldots+\ \ldots\ldots \text{ to infinity,}$$

the numerators being 1, 4, 6, 3, recurring.

12. If y denote the expression $ax+b$, show that the values which y assumes when values in arithmetical progression are substituted for x are themselves in arithmetical progression.

(*Afternoon.—Conics and Trigonometry.*)

13. If x_1y_1, x_2y_2, x_3y_3, be the Cartesian co-ordinates of three points P_1, P_2, P_3 in a plane, with respect to any pair of rectangular axes in the plane; determine, to the same axes, the equations of the parallel and of the perpendicular through the point P_3 to the right line $P_1 P_2$.

14. The equation of a circle, in rectangular Cartesian co-ordinates, being $x^2+y^2+2r\cos.\phi.x+2r\sin.\phi.y+r^2=0$; determine the co-ordinates of its centre, the square of its radius, and the common length of the two equal tangents to it from the origin.

15. Assuming the trigonometrical formulæ for sin. $(A+B)$ and for cos. $(A+B)$ in terms of the sines and cosines of A and B; deduce from them those for tan. $(A+B+C)$ and for cot. $(A+B+C)$ in terms of the tangents and cotangents of A, B, C, respectively.

16. Given, of a plane triangle, any two sides a and b, and the contained angle C; express, in terms of them, the remaining side c, the two remaining angles A and B, and the area triangle.

17. The measured lengths of the three sides a, b, c of a plane triangle are 13, 14, 15 feet, respectively; calculate, in square feet, its area, and, in ordinary fractions, the sines of its three angles A, B, C.

1878. *July* 17*th*.—Examiners, Prof. HENRICI, PH.D., F.R.S., and Rev. Prof. TOWNSEND, M.A., F.R.S.

1. Verify, by actual multiplication or otherwise, the algebraical identity

$$(y-z)^3+(z-x)^3+(x-y)^3=3(y-z)(z-x)(x-y).$$

2. Find, by actual multiplication or otherwise, and arrange in ascending powers of x supposed to be less than unity, the square of the progression

$$1+\tfrac{1}{2}x+\tfrac{1}{2}\,\tfrac{3}{4}x^2+\tfrac{1}{2}\,\tfrac{3}{4}\,\tfrac{5}{6}x^3+\tfrac{1}{2}\,\tfrac{3}{4}\,\tfrac{5}{6}\,\tfrac{7}{8}x^4+\&c., \text{ to infinity.}$$

3. Extract, by the ordinary process or otherwise, and arrange again in ascending powers of x supposed as before to be less than unity, the square root of the progression

$$1+x+x^2+x^3+x^4+\&c., \text{ to infinity.}$$

4. Given that, in an arithmetical progression of n terms commencing with unity, the sum is equal to the square of the number of terms: determine, by any method, the common difference.

5. Given that, in a geometrical progression of n terms commencing with unity, the sum is a number consisting of n digits each equal to unity; determine, by any method, the common ratio.

6. The sum of two numbers is 1878, and their product is 880821; determine them completely, each to the last integer figure.

7. Calculate, to the nearest halfpenny, the true present value of a bill for £152 8*s*., payable on December 31, and discounted on July 17, at $3\frac{1}{2}$ per cent. interest per annum.

8. The operatives in a factory being supposed to consist of a men, b women, c boys, and d girls; required the entire number of different combinations of p men, q women, r boys, and s girls, that can be told off from among them to any particular work.

9. Find the values of x, y, z which satisfy the three algebraical equations

$$ax+by+cz=p^2, \quad fx+gy+hz=q^2, \quad x^2+y^2+z^2=r^2.$$

10. Given that the two quadratic functions $a_1x^2+2h_1x+b_1$ and $a_2x^2+2h_2x+b_2$ have a common factor; prove, by any method, the equation of condition

$$(a_2b_2+a_2b_1-2h_1h_2)^2=4(h_1{}^2-a_1b_1)(h_2{}^2-a_2b_2).$$

11. Assuming the fundamental properties of logarithms, state and prove the ordinary rules by which the logarithm of a number not given in the tables, and conversely the number corresponding to a logarithm not given in the tables, may be calculated from the tables.

12. Given that, to ten decimal places, in the ordinary system, log. 2=0·30102999561, log. 3=0·4771212546; calculate, to the same number of places, the values of log. 4·5, of log. 6·75, and of log. 10·125, in the same system.

(*Afternoon.—Trigonometry and Conics.*)

13. Determine the sine and cosine of an angle whose tangent equals -2 and whose sign is positive.

14. A, B, C, being the angles in a triangle, show that

$$\tan. A \tan. B \tan. C=\tan. A+\tan. B+\tan. C.$$

15. A vertical pole (more than 100 feet high) consists of two parts, the lower being $\frac{1}{3}$ of the whole. From a point in a horizontal

plane through the foot of the pole, and 40 feet from it, the upper part subtends an angle whose tangent is equal to $\frac{1}{2}$. Determine the height of the pole.

16. Through the point of intersection of the straight lines whose equations are

$$2x-3y+7=0,$$
$$x+4y+3=0,$$

a straight line is drawn at right angles to the axis of x, and another at right angles to the line whose equation is

$$x+y+1=0.$$

Find the equations of these lines.

17. Find the equations of the tangents to the circle

$$x^2+y^2+2Ax+2By+C=0,$$

which are parallel to the line

$$x+2y-6=0.$$

18. Prove, by elementary or co-ordinate geometry, that the locus of points whose distances from two fixed points have a constant ratio, is a circle.

July 23*rd*.—Examiners, Dr. JOHN HOPKINSON, M.A., F.R.S., and Rev. Prof. TOWNSEND, M.A., F.R.S.

1. Multiply together $1\cdot3\dot{4}$ and $2\cdot5\dot{6}\dot{7}$, and extract the square root of the result correctly to seven significant figures.

2. At what rate of compound interest will a given sum be increased eleven-fold in 100 years?

$$[\log. 11=\frac{1\cdot0413927}{100}; \ \log. 1\cdot1266=0\cdot0517697;$$
$$\log. 1\cdot1267=0\cdot0518083.]$$

3. A man of 25 years of age can insure his life for £1000 by paying an annual premium of £18. Taking interest at 5 per cent. per annum, what will be an equitable composition for an annual subscription of £3?

4. Ten English labourers can do as much excavating in six days as nine French labourers in seven days; a Frenchman receives one franc per cubic metre; how many pence must an Englishman receive per cubic yard that his daily earnings may be 5 per cent. more than a Frenchman's?

[A metre may be taken as equal to $39\frac{3}{8}$ inches, and a franc to tenpence.]

5. The sum of the squares of two numbers is 1105, and their product is 552 times their difference. What are they?

6. There are m white men and n black men, n being greater than m. Find the number of ways in which each white man may have one black servant. If a white man may have any number of servants, in how many ways may every black man have a master?

7. Solve the equation:

$$\frac{x-a}{b}+\frac{x-b}{a}=\frac{b}{x-a}+\frac{a}{x-b}.$$

8. Under what conditions will x^3+ax^2+bx+c be divisible by x^2+px+q?

9. Sum the series $1+2x+3x^2+4x^3+\ .\ .\ .\ +nx^{n-1}$.

10. A sum of £1000 bearing interest at 5 per cent. is to be paid off in three annual instalments, the payments, including the interest due, to be the same each year, and the first payment to be due at the end of the first year. What must the yearly payment be?

11. Define a logarithm. Prove the rules of multiplication and division of numbers by the aid of logarithms. Wherein lies the convenience of our tables being calculated to base 10?

12. What are oranges a gross when 50 more for a sovereign lowers the price twopence a score?

(*Afternoon.—Trigonometry and Conics.*)

13. Given, for three trigonometrical angles α, β, γ, that tan. $\alpha=a$, tan. $\beta=b$, tan. $\gamma=c$; find the value of tan. $(\alpha+\beta+\gamma)$ in terms of a, b, c.

14. Given, of a rectilinear triangle, the base c, and the two base angles A and B; find, in terms of them, the two sides a and b, the altitude h, and the area of the triangle.

15. The co-ordinates (rectangular) of three points P, Q, R, in a plane being respectively (3, 4), (5, 6), (7, 8); determine, by any method, the area of the triangle PQR.

16. The equations, in rectangular co-ordinates, of a circle in a plane, and of a right line through its centre, being respectively $x^2+y^2=r^2$, and $x+y=0$; find, by any method, those of the two tangents to the circle which are parallel to the line.

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SOLUTIONS

TO THE FIRST

B.A. AND B.Sc.

PASS EXAMINATION PAPERS.

B.A. PASS SOLUTIONS.

1839. *May 27th.*

1. Art. 1.
2. (1) 3·141592. (2) ·0101.
3. (1) $x = 5$. (2) $x = -6$. (3) $x = 1$ or 3.

(4) $\sqrt{x^2 - 2x + 93} - \frac{x^2}{2} = 45 - x$

$$2\sqrt{x^2-2x+93} = x^2 - 2x + 90$$

$$(x^2-2x+93) - 2\sqrt{x^2-2x+93} - 3 = 0$$

$$\sqrt{x^2-2x+93} = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm 4}{2} = 3 \text{ or } -1$$

$$x^2 - 2x + 93 = 9 \qquad\qquad x^3 - 2x + 93 = 1$$

$$x^2 - 2x + 84 = 0 \qquad\qquad x^2 - 2x + 92 = 0$$

$$x = \frac{2 \pm \sqrt{4 - 336}}{2} \qquad\qquad x = \frac{2 \pm \sqrt{4 - 368}}{2}$$

$$= \frac{2 \pm \sqrt{-332}}{2} \qquad\qquad = \frac{2 \pm \sqrt{-364}}{2}$$

$$= 1 \pm \sqrt{-83} \qquad\qquad = 1 \pm \sqrt{-91}$$

$$x = 1 \pm \sqrt{-83} \text{ or } 1 \pm \sqrt{-91}. \quad \textit{Ans.}$$

4. Art. 232 (3).
5. Art. 234.
6. Art. 289.
7. Length of the other side =

$$\sqrt{100^2 - 42^2} = \sqrt{8236} = 90·75 \text{ nearly};$$

$$\therefore \text{area} = 90·75 \times 42 = 3811·5 \text{ sq. yds.}$$

8. (1) $\sqrt{x}+\sqrt{10+x}=\dfrac{20}{\sqrt{10+x}}$

$$\sqrt{10x+x^2}+10+x=20$$
$$\sqrt{10x+x^2}=10-x$$
$$10x+x^2=100-20x+x^2$$
$$30x=100$$
$$x=3\tfrac{1}{3}.$$

(2) $x^3+y^3=1001 \quad (1)$

$x+y = 11 \quad (2)$

From (2) $y=11-x$.

Substituting in (1)

$$x^3+1331-363x+33x^2-x^3=1001$$
$$33x^2-363x+330=0.$$
$$x^2-11x+10=0.$$
$$\left.\begin{aligned} x=\frac{11\pm\sqrt{121-40}}{2}=\frac{11\pm 9}{2}&=10 \text{ or } 1\\ y=11-x&=1 \text{ or } 10\end{aligned}\right\}.$$

Also see Ex. 6, p. 104.

9. Art. 254.

10. Art. 328, Eq. VI.

1840. *May 24th.*

1. Arts. 75, 76. ·00020402.
2. Arts. 95 . . . 98. 45·67.
3. £262 4*s.* $3\frac{1}{2}d.\frac{13}{20}$.
4. (1) $x=7$. (2) $x=3\pm\sqrt{-1}$.
   (3) $x=5$ or $11\frac{25}{28}$, $y=9$ or $4\frac{17}{42}$.
   (4) $\dfrac{252}{247}$ hrs. or 1 h. 1′ $12''\frac{216}{247}$.
5. (1) 425. (2) Here are two series:

$$\frac{2}{5}+\frac{2}{5^2}+\&c., \text{ and } \frac{3}{5^2}+\frac{3}{5^4}+\&c.\ldots$$

$$\text{Sum of 1st}=\frac{2}{5}\left\{\frac{1}{1-\frac{1}{25}}\right\}=\frac{2}{5}\times\frac{25}{24}=\frac{5}{12}$$

$$\text{,, 2nd}=\frac{3}{25}\left\{\frac{1}{1-\frac{1}{25}}\right\}=\frac{3}{25}\times\frac{25}{24}=\frac{1}{8}$$

$$\therefore \frac{5}{12}+\frac{1}{8}=\frac{13}{24}. \quad \textit{Ans.}$$

6. Art. 166. Substitute n for r in the formula $C_r=\&c.$
7. Art. 195, 4°, and Art. 190.
8. Art. 208 and for answer to (1) Art. 225. (2) Art. 226 (ϵ), and note to 141.

9. Art. 231.

(2) When $B = 0$, $\sin B = 0$, $\cos B = 1$,

$$\operatorname{Sin} \overline{A-B} = \sin (A-0) = \sin A.$$

When $A = B$, $\sin B = \sin A$, $\cos B = \cos A$,

$$\operatorname{Sin} \overline{A-B} = \sin 0 = \sin A \cos A - \sin A \cos A = 0.$$

When $A = \frac{\pi}{2}$, $\sin A = 1$, $\cos A = 0$,

$$\operatorname{Sin} (A-B) = \cos B - 0 = \cos B.$$

10. Art. 237.

11. (1) Art. 300, Eq. II. (2) The equation to the circle is

$$y^2 = 2\,a\,x - x^2 \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

Let θ = the inclination of the circle to the plane of projection,
x = the abscissa in the circle,
y = the ordinate „

Draw in the circle a radius (a) parallel to y. In the Orthographic projection

$$a \text{ is projected into } a \cos \theta = b \quad \ldots\ldots \quad (2)$$
$$y \quad „ \quad y \cos \theta = y' \quad \ldots\ldots \quad (3)$$

From (2) $\cos \theta = \frac{b}{a}$; substituting in (3)

$$y = \frac{y'}{\left(\frac{b}{a}\right)} \therefore y^2 = \frac{y'^2}{\left(\frac{b^2}{a^2}\right)}$$

$\therefore$ from (1) $y'^2 = \frac{b^2}{a^2}(2\,a\,x - x^2)$ the equation to an ellipse.

1841. *May* 31*st*.

1. (1) Arts. 107 and 111. (2) 104 men.
2. ·0348.
3. (1) Log 1323 = log 3^3 + log 7^2.

$$\begin{aligned} \log 3^3 &= 1{\cdot}4313639 \\ \log 7^2 &= 1{\cdot}6901960 \\ \hline \log 1323 &= 3{\cdot}1215599 \\ \log 1{\cdot}323 &= 0{\cdot}1215599 \end{aligned}$$

(2) Art 37.

4. Ex. 3, p. 98.
5. (1) $1 - 10\,b + 40\,b^2 - 80\,b^3 + 80\,b^4 - 32\,b^5$.

(2) $a^{\frac{3}{2}} + a^{\frac{1}{2}}\,b^{\frac{1}{2}} + a^{\frac{1}{2}}\,b + b$.

(3) $1 - x + x^2 - x^3 + x^4 - x^5 +$ &c. (4) $\frac{3\,(x+1)}{2\,(x^2 - x + 1)}$.

6. (1) $x = 12$. (2) $x = \pm\frac{1}{2}$. (3) $x = 17, y = 11$.

(4) $x = \pm\frac{\sqrt{5}}{2} - \frac{1}{2}, y = \frac{3}{2} \mp \frac{\sqrt{5}}{2}$. (5) $x = 13$ or -10.

(6) $x = \frac{5+3\sqrt{13}}{2}$ or $\frac{5+\sqrt{901}}{2}$, and see Ex. 6, p. 114, for solution.

7. $\frac{8.7.6.5.4.3.2.1}{1.2.1.2} = 10080$.

8. (1) 40 or 21. For solution, see Ex. 7, p. 59. (2) $1\frac{2391484}{4782969}$.

9. (1) Art. 233. (2) $\text{Tan}\ (45°+\beta) = \frac{\tan 45°+\tan\beta}{1-\tan 45°\tan\beta}$

$$= \frac{1+\tan\beta}{1-\tan\beta},$$

$$\text{Similarly, } \tan\ (45°-\beta) = \frac{1-\tan\beta}{1+\tan\beta}.$$

$$\therefore \tan\ (45°+\beta) - \tan\ (45°-\beta) = \frac{1+\tan\beta}{1-\tan\beta} - \frac{1-\tan\beta}{1+\tan\beta}$$

$$= \frac{4\tan\beta}{1-\tan^2\beta} = 2\tan 2\beta.$$

$$\therefore \tan\ (45°+\beta) = 2\tan 2\beta + \tan\ (45°-\beta).$$

10. Art. 235.

11. Art. 325.

1842. *October 3rd.*

1. Arts. 49, 55. £·39375.

2. (1) $43\frac{3}{5}$ miles. (2) £1766 10*s.* 6*d.*

3. (1) Ex. 4, p. 12. (2) $\frac{x-5}{x+5}$. (3) $\frac{1}{6}$.

4. (1) $x = 7$. (2) $x = 17, y = 3$. (3) $x = 3$ or $-\frac{1}{25}$.

(4) $x = 72$ or 8. (5) Ex. 6, p. 104.

(6) $16\frac{4}{11}$ min. past 12, and again at $49\frac{1}{11}$ min. past 12.

5. (1) Art. 195. 4° and 6°.

(2) $\text{Log}\ 10 = 1,\ \log 5 = \log\frac{10}{2},$

$$= \log 10 - \log 2 = ·6989700,$$

$$\log 2000 = \log 2 + \log 1000 = 3·3010300.$$

6. (1) Art. 208. (2) Art. 231.

7. (1) Art. 234. (2) $\frac{a}{b} = \frac{\sin 45°}{\sin 30°} = \frac{\frac{1}{2}\sqrt{2}}{\frac{1}{2}} = \frac{\sqrt{2}}{1},$

or $a : b :: \sqrt{2} : 1$. (3) Art. 242.

8. (1) Art. 237.

(2) $a = 3$, $b = 5$, $c = 7$; $s = 7{\cdot}5$, $s-a = 4{\cdot}5$, $s-b = 2{\cdot}5$, $s-c = {\cdot}5$.

$\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{42{\cdot}1875} = 6{\cdot}495$.

9. Art. 308.

1843. *October 2nd.*

1. (1) $\cdot\dot{0}58823529411764\dot{7}$. (2) ·008389522.
2. (1) Art. 189. (2) Art. 192.
3. (1) Arts. 23 . . . 25.

   (2) $32a^5 - 240a^4b + 720a^3b^2 - 1080a^2b^3 + 810ab^4 - 243b^5$.

   (3) $x - y$.
4. (1) Art. 46.

   (2) G. c. m. $= x^2 - 16x + 63$, lowest terms $\frac{x-1}{x+3}$.
5. (1) Art. 117. (2) 531440. (3) $\frac{1}{2}$. (4) 232.
6. (1) $x = 5$. (2) $x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}$, $y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$.

   (3) $x = \frac{21 \pm \sqrt{197}}{2}$, $y = \frac{21 \mp \sqrt{197}}{2}$.

   (4) $\sqrt{x^2 - 5x + 6} = 9$ or -2 $\therefore x = \frac{5 + 5\sqrt{13}}{2}$ or $\frac{5 \pm \sqrt{17}}{2}$. (5) 7, 21, 63.
7. Page 131, I. Example.
8. Arts. 231, 232.
9. (1) Art. 299, Eq. I. (2) Art. 300, Eq. II.

   (3) Art. 300, Eq. III.

1844. *October 7th.*

1. (1) ·0208. (2) $\cdot\dot{4}34782608695652173913\dot{0}$.

   (3) 6·8792838. (4) Art. 92.
2. £49 9*s*. 1·1320 . . . .
3. (1) $a^3 + 6a^2b + 12ab^2 + 8b^3 - 9a^2c - 36abc - 36b^2c + 27ac^2 + 54bc^2 - 27c^3$.

   (2) $a - b$. (3) $a - b + \frac{2b^2}{a} - \frac{2b^3}{a^2} + \frac{2b^4}{a^3} -$ &c.

   (4) The result indicates that it is impossible to separate $a^2 + b^2$ into rational factors containing the simple powers of a and b.

4. Art. 165, $P_r =$ &c., and make the necessary substitutions.

5. (1) Art. 114. (2) Art. 117. Examples proposed, 1. 2016.

2. 17th term is $\frac{30517578128}{43046721}$; sum $= 1772\frac{1120821}{215233605}$.

6. (1) $x = 13$.

(2) Transpose A_2 to the right side of the equation. Take half of A_1 the co-efficient of x, square and add it to both sides; this will make the left side a complete square. Extract the square root, and the result will be a simple equation, from which x may be found.

(3) Since $x = x_1 \quad \therefore x - x_1 = 0.$

$$\text{,,} \quad x = x_2 \quad \therefore x - x_2 = 0.$$

$$\therefore (x - x_1)(x - x_2) = 0,$$

$$\text{Or } x^2 - (x_1 + x_2)\, x + x_1 x_2 = 0.$$

Comparing this with $x^2 + A_1 x + A_2 = 0$,

$$\text{we have } x_1 + x_2 = -A_1\,;\ x_1 x_2 = A_2.$$

The roots of the equation, $x^2 + 6x - 55 = 0$, are 5 and -11, and $x^2 - (5 - 11)\, x - 55 = 0$.

(4) $x^{x+y} = y^{4a} \ldots (1) \qquad y^{x+y} = x^a \ldots (2)$

$$\therefore (x+y) \log x = 4a \log y \ldots (3)$$

$$a \log x = (x+y) \log y \ldots (4)$$

Divide (3) by (4) $\therefore \frac{x+y}{a} = \frac{4a}{x+y}$,

$$(x+y)^2 = 4a^2 \therefore x + y = 2a \ldots (5)$$

Substituting in (2) $x^a = y^{2a}$;

$$\text{take the } a\text{th root,} \quad \therefore x = y^2,$$

Substitute in (5) $\therefore y^2 + y = 2a$;

$$y = -\frac{1}{2} \pm \frac{1}{2}\sqrt{8a+1},$$

$$x = \frac{4a+1}{2} \mp \frac{1}{2}\sqrt{8a+1}.$$

7. (1) Arts. 208 and 233. (2) Tan $(A+B) = \tan(45^\circ + 30^\circ)$

$$= \frac{\sin(45^\circ + 30^\circ)}{\cos(45^\circ + 30^\circ)} = \frac{\frac{1}{2}\sqrt{2} \times \frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{2} \times \frac{1}{2}}{\frac{1}{2}\sqrt{2} \times \frac{1}{2}\sqrt{3} - \frac{1}{2}\sqrt{2} \times \frac{1}{2}}$$

$$= \frac{\frac{1}{2}\sqrt{3} + \frac{1}{2}}{\frac{1}{2}\sqrt{3} - \frac{1}{2}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2 + \sqrt{3} = \tan 75^\circ.$$

8. (1) Art. 236. (2) Arts. 257 and 258. (3) Art. 260.

9. Art. 322, or 325.

1845. *October* 30*th*.

1. (1) ·75. (2) Art. 88. (3) Art. 88. Note.
2. (1) £138 17*s*. $9\frac{1}{3}$*d*.; £236 2*s*. $2\frac{2}{3}$*d*. (2) £67 6*s*. $8\frac{3}{4}$*d*.
3. (1) Arts. 23 . . . 26.
   (2) $x^3+(a+b+c)\,x^2+(a\,b+a\,c+b\,c)\,x+a\,b\,c$.
   (3) If $a=b=c$, this becomes $x^3+3\,a\,x^2+3\,a^2\,x+a^3$.
   (4) If $b=-a$, and $c=0$, it becomes $x^3-a^2\,x$. (5) $x-5$.
4. (1) 1, $2\frac{1}{2}$, 4, $5\frac{1}{2}$, 7.
   (2) (1) $x=\frac{3}{4}$, (2) $x=7$; $y=9$. (3) $x=-17$ or $+3$.
   (3) Arts. 177, 178.
5. (1) Art. 165. (2) Ex. 1, p. 94. (3) Art. 165 (III.)
6. (1) Art. 189. (2) Art. 193.
   (3) Log 45 = 1·6532125,
   log 450 = 2·6532125,
   log 4·5 = 0·6532125.
   (4) Art. 192. (5) Art. 195.
7. (1) Art. 211. (2) Arts. 217, 218. (3) Art. 231.
8. Art. 237.
9. (1) Art. 294. (2) Art. 308.

1846. *October* 26*th*.

1. (1) $\frac{9}{5650}$. (2) $\cdot 07692\dot{3}$.

(3) Equal numbers are such as, being subtracted one from the other, leave no remainder: or such as, being divided one by the other, give the quotient 1.

2. Ex. 2, p. 98.
3. (1) 3102. (2) 310·2. (3) Arts. 96, 97.
4. By the question we have,

	Men.	S.	Mo.	
For officers	4 ×	40 ×	6 =	960
„ midshipmen	12 ×	30 ×	6 =	2160
„ sailors	110 ×	22 ×	3 =	7260

Adding, 7260 + 2160 + 960 = 10380;

			£	s.	d.	
∴ 10380 :	960 ::	1000 :	92	9	$8\frac{1}{2}\ \frac{132}{1038}$	officers' s.
10380 :	2160 ::	1000 :	208	1	$10\frac{0}{4}\ \frac{816}{1038}$	middies' s.
10380 :	7260 ::	1000 :	699	8	$5\frac{1}{4}\ \frac{90}{1038}$	sailors' s.

Proof,—officers' + middies' + sailors' = £1000.

5. (1) Art. 75. (2) Art. 101.
(3) (1) Art. 145. (2) Art. 147.

6. (1) Art. 114. (2) Art. 117.
(α) $x = 1$. (β) $x = 7$; $y = 11$.
(γ) $x = 15$ or 7; $y = 7$ or 15. (δ) $x^2 - 5x + 6 = 0$.

7. Art. 166 (III).

8. Art. 233.

9. Art. 237.

10. (1) Art. 282, Def. (2) Art. 289.

11. Art. 325.

1847. *October 25th.*

1. £2033 4*s.* $0\frac{1}{4}$*d.* $\frac{181}{203}$.

2. 948·1.

3. (1) $a^{n-1} + a^{n-2} b + a^{n-3} b^2 \ldots\ldots\ a^2 b^{n-3} + a b^{n-2} + b^{n-1}$.

(2) $\dfrac{x + x^2 - x^4}{1 + x - x^3 - x^4}$.

(3) When n is negative, it indicates the reciprocal of the positive expression: so $a^{-n} = \dfrac{1}{a^n}$. When n is fractional (suppose $= \dfrac{p}{q}$) it means, take the qth root of the pth power: so if $n = \dfrac{p}{q}$, $a^n = \sqrt[q]{a^p}$.

4. (1) Art. 146.
(2) If $a : b :: c : d$

$$\therefore \frac{a}{b} = \frac{c}{d} \quad \therefore \frac{a^2}{b^2} = \frac{c^2}{d^2}$$

$$(+1) \qquad \frac{a^2}{b^2} + 1 = \frac{c^2}{d^2} + 1, \text{ or } \frac{a^2 + b^2}{b^2} = \frac{c^2 + d^2}{d^2} \ \ldots\ldots \ (a)$$

$$(-1) \qquad \frac{a^2}{b^2} - 1 = \frac{c^2}{d^2} - 1, \text{ or } \frac{a^2 - b^2}{b^2} = \frac{c^2 - d^2}{d^2} \ \ldots\ldots \ (b)$$

$$(a) \div (b) \qquad \frac{a^2 + b^2}{a^2 - b^2} = \frac{c^2 + d^2}{c^2 - d^2}.$$

$$\therefore a^2 + b^2 : a^2 - b^2 :: c^2 + d^2 : c^2 - d^2.$$

5. Ex. near top of p. 94.

6. (α) $x = 6$. (β) $x = 3$; $y = 2$.
(γ) $x = 11$ or 5; $y = 5$ or 11. (δ) $x = 8$ or 4.

7. Art. 195, 6°.

8. (1) Arts. 231, 232. (2) From Art. 233 (4) and (5).
$\sin 3A = 3\sin A - 4\sin^3 A$, $\cos 3A = 4\cos^3 A - 3\cos A$.

Now $\sin 36° = \cos 54°$, or $\sin(2\times 18°) = \cos(3\times 18°)$;

$\therefore \sin 36° = 4\cos^3 18° - 3\cos 18°$.

But $\sin 36° = 2\sin 18°\cos 18° \ldots\ldots\ldots (a)$

$\therefore 2\sin 18° = 4\cos^2 18° - 3$;

and reducing and transposing,

$$4\sin^2 18° + 2\sin 18° = 1;$$

$$\text{whence } \sin 18° = \frac{\sqrt{5}-1}{4}.$$

$$\text{and } \cos 18° = \frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}}.$$

which substituted in (a) give

$$\sin 36° = \frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}};\quad \cos 36° = \sqrt{\frac{3+\sqrt{5}}{8}} = \frac{\sqrt{5}+1}{4}.$$

9. (1) Art. 281.

(2) Since the line passes through the points (x', y') and $\left(0, \frac{y'}{2}\right)$ we have (see Art. 291 (a)), substituting a for m,

$$y' = a\,x' + b \ldots\ldots\ldots\ldots (1)$$

$$\frac{y'}{2} = a\times 0 + b \ldots\ldots\ldots\ldots (2)$$

From (2) we have ... $b = \frac{y'}{2}$.

Substituting in (1) we get $a = \frac{y'}{2x'}$.

10. Art. 308.

1848. *October 23rd.*

1. (1) Art. 37, § 2. (2) Art. 37, § 4. (3) $2^8\times 3^8 = 1679616$.
2. (1) Art. 96. (2) $\sqrt{2\frac{3}{4}} = 1{\cdot}65831$.
3. £32 4*s*. 0$\frac{3}{4}$*d*.
4. (1) $a\,x + \frac{1}{4}\left(\frac{a^3}{x^3} - \frac{x^3}{a^3}\right) - \frac{1}{2}\left(\frac{a^2}{x} - \frac{x^2}{a}\right) + \frac{1}{8}\left(\frac{a}{x} + \frac{x}{a}\right) - \frac{5}{16}$.

   (2) Ex. 9, p. 31. (3) Arts. 23 ... 25.
5. (α) $x = 35$. (β) $x = 7$; $y = 9$.

   (γ) $x = \pm\sqrt{\left(\frac{-q\pm\sqrt{q^2-4s}}{2}\right)}$.

   (4) No. of terms, 5 or 10.

6. Art. 194.
7. (1) Art. 211. (2) Art. 218. (3) Art. 234.
8. Art. 237.
9. (1) Art. 299. (2) Art. 300, Eq. II. (3) Art. 300, Eq. III.
(Repetition of Quest. 9, Oct. 5, 1843.)

1849. *October 22nd.*

1. (1) Art. 75. (2) Art. 77. (3) ·0288. (4) ·00096875.
2. 72 miles.
3. (1) and (2) Art. 193.
   (3) Log 80 = 1·9030900
   log 81 = 1·9084852
   log 360 = 2·5563026
   (4) Conclusion of Art. 196.
4. (1) a^6-b^6. (2) $\sqrt{a}+\sqrt{b}$. (3) $a_{\frac{5}{12}}$. (4) $(c-d)\sqrt{a\,x}$.
5. Ex. 4, p. 59.
   (α) $x=7$. (β) $x=8\frac{1}{2}$; $y=9$; $z=9\frac{1}{2}$.
   (γ) $x=\dfrac{n\pm\sqrt{2\,m-n^2}}{2}$; $y=\dfrac{n\mp\sqrt{2\,m-n^2}}{2}$.
   (5) In this form $x=\dfrac{a+\sqrt{a^2-4\,b}}{2}$ or $\dfrac{a-\sqrt{a^2-4\,b}}{2}$,

$$\text{whose sum} = \frac{2\,a}{2}, \text{ or } a,$$

$$\text{and product} = \frac{a^2-(a^2-4\,b)}{4} = \frac{4\,b}{4}, \text{ or } b.$$

6. (1) Arts. 211, 217. (2) Art. 231.
7. Art. 237.
8. (1) Art. 292, β. (2) Art. 325, Eq. II.

1850. *October 28th.*

1. (1) Arts. 95, 96. (2) 1·41421.
2. The third pipe conveys x gallons per minute.
   „ first „ $x+10$ „
   „ second „ $x-5$ „
   ∴ The three pipes convey $(3\,x+5)$ galls. in 1 minute.

$$\text{But } 3\,x+5=\frac{820}{20}=41 \text{ by the question.}$$

∴ $x=12$, the number of galls. conveyed through the third pipe
also 22 „ „ „ first
and 7 „ „ „ second.

3. (1) $1+\frac{5x^2}{12}-\frac{5x^4}{36}-\frac{x^6}{16}$.

(2) Repetition of Quest. 3, (1) and (3), 1847.

(3) Ex. 11, p. 32.

4. (1) 1 and 49.

(2) (α) $x=9$. (β) $x=11$ or -31.

(γ) $x=19$ or 17; $y=17$ or 19.

(δ) $x=\frac{\log b_1 \log c-\log b \log c_1}{\log a \log b_1-\log a_1 \log b}$,

$$y=\frac{\log a \log c_1-\log a_1 \log c}{\log a \log b_1-\log a_1 \log b}.$$

(3) If $a x^2+b x+b=0$, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Put $\frac{-b+\sqrt{b^2-4ac}}{2a}=x_1$, and $\frac{-b-\sqrt{b^2-4ac}}{2a}=x_2$.

$$\frac{x_1}{x_2}+\frac{x_2}{x_1}=\frac{x_1^2+x_2^2}{x_1 x_2}.$$

$$x_1^2=\frac{2b^2-4ac-2b\sqrt{b^2-4ac}}{4a^2};\ x_2^2=\frac{2b^2-4ac+2b\sqrt{b^2-4ac}}{4a^2}.$$

$$x_1 x_2=\frac{b^2-(b^2-4ac)}{4a^2}=\frac{4ac}{4a^2}.$$

$$\frac{x_1}{x_2}+\frac{x_2}{x_1}=\frac{4b^2-8ac}{4ac}=\frac{b^2-2ac}{ac}.$$

5. 230300 nights, any one man 4606, Ex. 3, p. 95.

6. (1) Art. 211. (2) Art. 217. (3) Art. 234.

7. (1) Art. 237. (2) Area $=\sqrt{6\,.\,3\,.\,2\cdot 1}=\sqrt{36}=6$.

8. (1) Here tan of the angle between the given line and the axis of x is $-\frac{b}{a}$; the negative reciprocal to which is $\frac{a}{b}$, and as this is the only condition to be satisfied, the equation required is,

$$\frac{y}{a}-\frac{x}{b}=p,$$

where p may be taken at pleasure.

(2) Art. 308. (3) Art. 311. (4) Art. 312.

1851. *October 27th.*

1. Ex. (1), p. 87.

2. Arts. 23 . . . 25. [1] a^4-b^4. [2] $a^4+a^2b^2+b^4$.

Performing the division we have

$$\begin{array}{l} x+y)\ x^2+a\,x+b\ (x+(a-y) \\ \quad\ \ \underline{x^2+x\,y\qquad\quad} \\ \qquad\qquad x\,(a-y)+b \\ \qquad\qquad \underline{x\,(a-y)+y\,(a-y)} \end{array}$$

$$\therefore y\,(a-y)=b, \text{ or } y^2-a\,y=-b,$$

$$\text{whence } y=\frac{a\pm\sqrt{a^2-4\,b}}{2},$$

and $x^2+a\,x+b$ is divisible by $x+y$, when y has the value just found.

3. (1) Common diff. $=\dfrac{b-a}{n+1}$. The means will therefore be

$$\frac{n\,a+b}{n+1},\ \frac{(n-1)\,a+2\,b}{n+1},\ \frac{(n-2)\,a+3\,b}{n+1}\ \ldots\ldots\ldots$$

$$\frac{2\,a+(n-1)\,b}{n+1},\ \frac{a+n\,b}{n+1}.$$

(2) $1+\dfrac{1}{2}+\dfrac{1}{4}+$ *ad inf.* $=2$. (See Ex. (11), p. 63.)

Equations (α) $x=1$. (β) $x=17$; $y=2$.

(γ) $x=17$ or 1; $y=1$ or 17.

(δ) $xy=\dfrac{a^3-b}{3\,a}$;

$$x=\frac{a}{2}\pm\frac{1}{2}\sqrt{\frac{4\,b-a^3}{3\,a}};\ y=\frac{a}{2}\mp\frac{1}{2}\sqrt{\frac{4\,b-a^3}{3\,a}}.$$

(3) P. 94, Ex. 2.

4. (1) Art. 189. (2) and (3) Arts. 191, 192.

(4) Log $21=\log 3+\log 7$,

$\log 210=\log 3+\log 7+\log 10$.

$$\log 2{\cdot}1=\log\frac{21}{10}=\log 3+\log 7-\log 10.$$

$$\therefore \log 21=1{\cdot}3222193,$$
$$\log 210=2{\cdot}3222193,$$
$$\log 2{\cdot}1=0{\cdot}3222193.$$

5. (1) Art. 216. (2) Arts. 211 and 217.

6. Arts. 231, 232.

7. Art. 237.

8. (1) General definition, p. 209.

(2) The constant ratio is called the *Eccentricity*; and a *Conic* is called a *Parabola*, an *Ellipse*, or an *Hyperbola*, according as the eccentricity is equal to, less than, or greater than, unity.

(3) Yes, see Art. 326. (4) Art. 325, Eq. II.

1852. *October 25th.*

1. (1) $\cdot\dot{1}4285\dot{7}$, (2) and (3). Repetition of Quest. 1, (2) and (3), 1845.

2. Let x = the rate of marching of the one party, (miles.)

$x+\frac{1}{4}=$ „ „ other.

But $\dfrac{39}{x}-\dfrac{39}{x+\frac{1}{4}}=1$ hour by the question;

$$\therefore x=3,$$

and the rates of marching are 3 and $3\frac{1}{4}$ miles respectively per hour.

3. (1) When n is an odd integer.

(2) No. (3) $\dfrac{x^2+1}{(x+1)^3\,(x+3)}$.

4. (α) $x=6$. (β) $x=3$; $y=7$.

(γ) $x=11$ or 7 : $y=7$ or 11.

(δ) Since $x+y : x-y :: a : b$

$$x : y :: a+b : a-b$$

$$\therefore x=\frac{a+b}{a-b}\cdot y.$$

Substituting in the second equation we have

$$\left(\frac{a+b}{a-b}\right)^2 y^2-y^2=c$$

$$\left\{\left(\frac{a+b}{a-b}\right)^2-1\right\}y^2=c$$

$$\frac{4\,a\,b}{(a-b)^2}\,y^2=c,$$

$$\text{whence } y=\frac{a-b}{2}\sqrt{\frac{c}{ab}},$$

$$x=\frac{a+b}{2}\sqrt{\frac{c}{ab}}.$$

The required numbers are, 6, 8, 10, 12.

5. (1) Art. 165 (I). (2) Ex. 4, p. 95.

6. (1) Arts. 231, 232. (2) Art. 243.

7. Art. 237.

8. (1) Art. 294. (2) Art. 342 or 345.

1853. *October 25th.*

1. (1) Arts. 155, 159. (2) Ex. 2, p. 87. (3) Art. 161.
(4) Ex. 7, p. 91.

2. After the crew had been at sea 11 days the provisions would have lasted them 10 days longer; but they last only 5 days; therefore the number of persons on board after picking up the party from the wreck must have been double what it was before, or the party from the wreck = the crew = 26.

3. (1) Art. 189. (2) and (3) Art. 195, 1° and 2°.
(4) Art. 192.
(5) $\text{Log } 360 = \log 2^2 + \log 3^2 + \log 10.$
$= 2 \log 2 + 2 \log 3 + 1.$
$\log 36 = 2 \log 2 + 2 \log 3.$
$$\log 3{\cdot}6 = \log \frac{36}{10} = 2 \log 2 + 2 \log 3 - 1.$$

4. (1) $x^4 + (a+b+c+d)\, x^3 + (a\,b + a\,c + a\,d + b\,c + b\,d + c\,d)\, x^2 + (a\,b\,c + a\,b\,d + a\,c\,d + b\,c\,d)\, x + a\,b\,c\,d.$
(2) $x^4 + 4\,a\,x^3 + 6\,a^2\,x^2 + 4\,a^3\,x + a^4.$
(3) $\sqrt{m} + \sqrt{n}.$ (4) $a^2 - a\,b + b^2.$ (5) $\dfrac{a^{\frac{1}{2}}}{a^{-\frac{1}{2}}} = a^{\frac{1}{2}+\frac{1}{2}} = a.$
(6) Art. 82.

5. (α) $x = 12.$ (β) $x = 11$; $y = 7.$
(γ) $x = \dfrac{\sqrt{2\,m - n^2} + n}{2}$; $y = \dfrac{\sqrt{2\,m - n^2} - n}{2}.$

6. (1) $\text{Sin}\,(A - B) = \sin\,\{A + (-B)\}$
$= \sin A \cos\,(-B) + \cos A \sin\,(-B)$
$= \sin A \cos B - \cos A \sin B.$

Again, $\cos\,(A+B) = \sin\,\{(90 - A) + (-B)\}$
$= \sin\,(90 - A) \cos\,(-B) + \cos\,(90 - A) \sin\,(-B)$
$= \cos A \cos B - \sin A \sin B.$

Also, $\cos\,(A - B) = \sin\,\{(90 - A) + B\}$
$= \sin\,(90 - A) \cos B + \cos\,(90 - A) \sin B$
$= \cos A \cos B + \sin A \sin B.$

(2) $\frac{1}{2}$, see Art. 227.

7. Art. 234.

8. (1) Art. 295. (2) Arts. 321, 322.

1854. *October 23rd.*

1. (1) Ex. 3, p. 87. (2) £3,574 4*s*. $10\frac{1}{2}$*d*.

2. One hour = 3600 seconds; and $3600 \times 2 \times 28 = 201600$, the number of inches marched in one hour.

But 201600 inches $= 3\cdot\dot{1}\dot{8}$ miles $= 3\frac{2}{11}$ miles, the rate per hour of marching. Again,

$$\frac{20}{3\frac{2}{11}} = \frac{220}{35} = \frac{44}{7} = 6\tfrac{2}{7} \text{ hours.}$$

$$\therefore (6\tfrac{2}{7}+1) \text{ hours} = 7\tfrac{2}{7} \text{ hours} = 7 \text{ hrs. } 17\tfrac{1}{7} \text{ min.,}$$

the time they will take to reach the garrison.

3. (1) Arts. 23, 24, 25. (2) Art. 82.

(3) x^n+y^n is divisible by $x+y$ when n is *odd*, but not when n is *even*.

PROOF: Assume $a = x+y$,

$$\therefore x = a-y,$$

$$\text{and } x^n = a^n - n\,a^{n-1}\,y + \ldots\ldots \pm n\,a\,y^{n-1} \mp y^n,$$

where y^n is negative, if n be odd.

Transpose y^n,

$$\therefore x^n + y^n = a^n - n\,a^{n-1}\,y + \ldots\ldots + n\,a\,y^{n-1},$$

if n be odd, which is divisible by a or $x+y$.

Secondly, let n be *even*,

$$\text{then } x^n = a^n - n\,a^{n-1}\,y + \ldots\ldots - n\,a\,y^{n-1} + y^n.$$

Add y^n to both sides,

$$\therefore x^n + y^n = a^n - n\,a^{n-1}\,y + \ldots\ldots - n\,a\,y^{n-1} + 2\,y^n,$$

which is not divisible by a or $x+y$ without remainder.

(4) x^n+y^n is never divisible by $x-y$.

$$\frac{x^n+y^n}{x-y} = \frac{x^n-y^n+2\,y^n}{x-y} = \frac{x^n-y^n}{x-y} + \frac{2\,y^n}{x-y}$$

$$= \text{a whole number and a fraction.}$$

$\therefore x^n+y^n$ is never divisible by $x-y$.

To show that $\dfrac{x^n-y^n}{x-y}$ is a whole number, or that x^n-y^n is divisible by $x-y$ without remainder, whatever positive integer n may be,—

$$\text{Let } x-y = b,$$

$$\therefore x = b+y,$$

$$x^n = b^n + n\,b^{n-1}\,y + \ldots\ldots + n\,b\,y^{n-1} + y^n,$$

Transpose y^n,

$$\therefore x^n - y^n = b^n + n\,b^{n-1}\,y + \ldots\ldots + n\,b\,y^{n-1}.$$

Now, since every term is divisible by b or $x-y$ without remainder, $\therefore x^n-y^n$ is divisible by $x-y$ without remainder, whatever n may be.

4. (1) Art. 165 (II). (2) 3360.

5. (1) Let $x =$ the number required,

$$\text{then } \frac{x}{3}-\frac{x}{4} = 16 \text{ by the question,}$$

$$\text{or } 4x-3x = 192,$$

$$\therefore x = 192.$$

(2)
$$xy(x^2+y^2) = 3 \ldots\ldots\ldots\ldots A$$
$$x^2y^2(x^4+y^4) = 7 \ldots\ldots\ldots\ldots B$$

$$A^2 \text{ is } x^2y^2(x^4+2x^2y^2+y^4) = 9$$
$$\text{or } x^2y^2(x^4+y^4)+2x^4y^4 = 9$$
$$\text{But } B \text{ is } x^2y^2(x^4+y^4) = 7$$
$$\text{Subtracting} \quad 2x^4y^4 = 2$$
$$x^4y^4 = 1$$
$$xy = 1.$$

Substituting this value of xy in A, we get

$$x^2+y^2 = 3$$
$$\text{But } 2xy = 2$$
$$\text{Adding } x^2+2xy+y^2 = 5$$
$$\text{or } (x+y)^2 = 5$$
$$\therefore x+y = \sqrt{5}$$
$$\text{Again, } x^2+y^2 = 3$$
$$\text{and } -2xy = -2$$
$$\text{Adding } x^2-2xy+y^2 = 1$$
$$\text{or } (x-y)^2 = 1$$
$$\therefore x-y = 1.$$
$$\text{But } x+y = \sqrt{5}.$$
$$\text{Adding } 2x = \sqrt{5}+1$$
$$x = \frac{\sqrt{5}+1}{2}$$

Subtracting $(x-y)$ from $(x+y)$ we get

$$2y = \sqrt{5}-1, \therefore y = \frac{\sqrt{5}-1}{2}.$$

(3) Let $x-d$, x, $x+d$, represent the three numbers, (d being the common difference,)

$$\therefore (x-d)^2+x^2+(x+d)^2 = 93 \ldots\ldots\ldots A$$
$$3\,(x-d)+4\,x+5\,(x+d) = 66 \ldots\ldots\ldots B$$
$$\text{From } A,\ 3\,x^2+2\,d^2 = 93 \ldots\ldots\ldots C$$
$$\text{,, } \ B,\ 12\,x+2\,d = 66$$

$$\text{Whence } d = 33-6\,x, \quad d^2 = (33-6\,x)^2.$$

Substituting in C, we get

$$3\,x^2+2\,(33-6\,x)^2 = 93,$$

$$\text{Whence } x^2 - \frac{792}{75}\,x = -\frac{417}{15}$$

$$x = \frac{396}{75} \pm \sqrt{\frac{156816}{5625} - \frac{156375}{5625}}$$

$$x = \frac{396}{75} \pm \frac{21}{75} = 5\,\frac{14}{25} \text{ and } 5.$$

$$\text{But } d = 33-6\,x.$$

Taking $x = 5$, we get $d = 3$; and the numbers are 2, 5, 8.

,, $5\frac{14}{25}$,, $= -\frac{9}{25}$,, $5\frac{23}{25}$, $5\frac{14}{25}$, $5\frac{1}{5}$.

6. Art. 231.
7. Art. 237.
8. (1) Art. 288. (2) Arts. 341, 342.

1855. *October 24th.*

1. (1) $\cdot 14285\dot{7}$.

(2) and (3) Art. 88. Repetition of Questions B.A. Ex. 1845, Qu. 1 and 1852, Qu. 1.

2. (1) *Involution* is the process of multiplying a quantity by itself any number of times.

Evolution is the finding of a quantity which being multiplied into itself a given number of times shall become equal to a given quantity; in other words, Evolution is the extraction of any *root* of a quantity.

(2)
$$(x+a)\,(x+b)\,(x+c) =$$
$$x^3+(a+b+c)\,x^2+(a\,b+a\,c+b\,c)\,x+a\,b\,c,$$

which, when $a = b = c$, becomes

$$x^3+3\,a\,x^2+3\,a^2\,x+a^3.$$

(3) Art. 94. To extract the square root of an algebraical expression, the rule is identical with that for the square root of

numbers, except that the new term is always found by dividing the *first* term of the remainder by the *first* term of the divisor.

$$
\begin{array}{l|l}
 & a^4-2a^3x+3a^2x^2-2ax^3+x^4\ (a^2-ax+x^2\quad Ans. \\
 & a^4 \\
2a^2-ax & -2a^3x+3a^2x^2 \\
 & -2a^3x+\ \ a^2x^2 \\
2a^2-2ax+x^2 & \qquad 2a^2x^2-2ax^3+x^4 \\
 & \qquad 2a^2x^2-2ax^3+x^4
\end{array}
$$

(4) Art. 96.

3. (1) $x=7$. (2) $x=11$, $y=7$.

(3) Squaring the first equation, we have

$$x^2+2xy+y^2=a^2$$
$$x^2+mxy+y^2=b$$
$$(2-m)\,xy=a^2-b$$
$$xy=\frac{a^2-b}{2-m}$$
$$4xy=\frac{4(a^2-b)}{2-m}.$$

Subtracting this from $x^2+2xy+y^2=a^2$ we get

$$x^2-2xy+y^2=a^2-\frac{4(a^2-b)}{2-m},$$

Or, $$(x-y)^2=\frac{4b-(2+m)a^2}{2-m};$$

$$x-y=\sqrt{\frac{4b-(2+m)a^2}{2-m}};$$

But $x+y=a$.

By addition $x=\frac{1}{2}\left(a+\sqrt{\frac{4b-(2+m)a^2}{2-m}}\right)$

By subtraction $y=\frac{1}{2}\left(a-\sqrt{\frac{4b-(2+m)a^2}{2-m}}\right)$

(4) $yzu=a^3$ . . . . (A),
$xzu=b^3$ . . . . (B),
$xyu=c^3$ . . . . (C),
$xyz=d^3$ . . . . (D).

Multiplying all the equations together, $x^3y^3z^3u^3=a^3b^3c^3d^3$.

$\therefore xyzu=abcd$. . . . . (E);

Dividing (E) by (A), we get

$$\frac{x\,y\,z\,u}{y\,z\,u} = \frac{a\,b\,c\,d}{a^3};$$

whence $$x = \frac{b\,c\,d}{a^2}.$$

Similarly.

$$\frac{(E)}{(B)} \text{ gives } y = \frac{a\,c\,d}{b^2};$$

$$\frac{(E)}{(C)} \quad \text{,,} \quad z = \frac{a\,b\,d}{c^2};$$

$$\frac{(E)}{(D)} \quad \text{,,} \quad u = \frac{a\,b\,c}{d^2}.$$

(5) Art. 182.

4. (1) Art. 195. (2) Art. 192.

5. (1) Art. 199. (2) and (3) Art. 217.

6. Art. 254.

7. Art. 299. [1] and [2] Art. 300. (3) Art. 325.

(4) The centre being the origin, the value of x, when $y=0$, gives one semi-axe, and the value of y, when $x=0$, the other. Hence in

$$a^2\,x^2 + b^2\,y^2 = c^4,$$

$$\text{Let } y = 0, \qquad \therefore \quad x = \frac{c^2}{a};$$

$$\text{,,} \quad x = 0, \qquad \therefore \quad y = \frac{c^2}{b};$$

$$\therefore\ \frac{c^2}{a},\ \frac{c^2}{b}, \text{ are the semi-axes.}$$

1856. *October 29th.*

1. Here, by the question, 12 men in 11 days complete 220 yards; and it is required to find how many men will complete (700 − 220) 480 yards, in (29 − 11) 18 days. Hence

$$\left.\begin{array}{r}220 \text{ yards}\\ 18 \text{ days}\end{array}\right\} : \left.\begin{array}{r}480 \text{ yards}\\ 11 \text{ days}\end{array}\right\} :: \overset{\text{Men.}}{12} : \overset{\text{Men.}}{16}$$

That is, 16 men will complete the remaining 480 yards in the time proposed. Therefore 4 men must be added to the former 12.

2. Art. 82.

(2) $\sqrt{4a^4+12a^3x+13a^2x^2+6ax^3+x^4} = 2a^2+3ax+x^2$.

(3) Since $\frac{1}{2} = \frac{3}{6}$, $\therefore \quad 3^{\frac{1}{2}} = 3^{\frac{3}{6}} = 27^{\frac{1}{6}}$;

Also, $\frac{1}{3} = \frac{2}{6}$, $\therefore \quad 2^{\frac{1}{3}} = 2^{\frac{2}{6}} = 4^{\frac{1}{6}}$.

3. (α) $x = 12$. (β) $x = 15$, $y = 3$.

(γ) From the second equation, $x = n - by$;

Substituting in the first,

$$(n-by)^2 + ay(n-by) + y^2 = m.$$

Expanding and arranging the terms,

$$(b^2-ab+1)y^2 + (a-2b)n \cdot y = m-n^2;$$

$$\text{whence } y^2 + \frac{(a-2b)n}{b^2-ab+1}y = \frac{m-n^2}{b^2-ab+1}.$$

Solving the quadratic,

$$y = \frac{(2b-a)n \pm \sqrt{4m(b^2-ab+1) + (a^2-4)n^2}}{2(b^2-ab+1)}.$$

$$\text{But } x = n - by;$$

$$\therefore x = \frac{(2-ab)n \mp b\sqrt{4m(b^2-ab+1) + (a^2-4)n^2}}{2(b^2-ab+1)}.$$

If in the first of the equations (γ) we take $a = 0$, the expressions for x and y become, when $b = 1$,

$$x = \tfrac{1}{2}(n \mp \sqrt{2m-n^2}),$$
$$y = \tfrac{1}{2}(n \pm \sqrt{2m-n^2}).$$

Also, when $b = -1$,

$$x = \tfrac{1}{2}(n \pm \sqrt{2m-n^2}),$$
$$y = \tfrac{1}{2}(-n \pm \sqrt{2m-n^2}).$$

When $a = 2$, $b = 1$, the equations become

$$x^2+2xy+y^2 = m$$
$$x+y = n.$$

Here, since $x^2+2xy+y^2$ is the square of $x+y$, the equations are not *independent*, and are consequently insufficient to determine the values of x and y, the only conclusion deducible being that $m = n^2$.

(δ) $$\left(\frac{x}{a}\right)^a \left(\frac{y}{b}\right)^b = c \quad . \quad . \quad . \quad . \ (1),$$

$$\left(\frac{x}{b}\right)^b \left(\frac{y}{a}\right)^a = d \quad . \quad . \quad . \quad . \ (2),$$

From (1) $$\left(\frac{x}{a}\right)^a = \frac{c\, b^b}{y^b}; \qquad \therefore \ \frac{x}{a} = \frac{c^{\frac{1}{a}}\, b^{\frac{b}{a}}}{y^{\frac{b}{a}}};$$

$$\text{whence } x = \frac{a\, c^{\frac{1}{a}}\, b^{\frac{b}{a}}}{y^{\frac{b}{a}}} \quad . \quad . \ (3).$$

From (2) $$\left(\frac{x}{b}\right)^b = \frac{d\, a^a}{y^a}; \qquad \therefore \ \frac{x}{b} = \frac{d^{\frac{1}{b}}\, a^{\frac{a}{b}}}{y^{\frac{a}{b}}};$$

$$\text{and } x = \frac{b\, d^{\frac{1}{b}}\, a^{\frac{a}{b}}}{y^{\frac{a}{b}}} \quad . \quad . \quad . \ (4).$$

Equating (3) and (4), also inverting the values of x, we get

$$\frac{y^{\frac{b}{a}}}{a\, c^{\frac{1}{a}}\, b^{\frac{b}{a}}} = \frac{y^{\frac{a}{b}}}{b\, d^{\frac{1}{b}}\, a^{\frac{a}{b}}};$$

Dividing by $y^{\frac{b}{a}}$, we have

$$\frac{1}{a\, c^{\frac{1}{a}}\, b^{\frac{b}{a}}} = \frac{y^{\frac{a}{b}-\frac{b}{a}}}{b\, d^{\frac{1}{b}}\, a^{\frac{a}{b}}};$$

$$\therefore \ y^{\frac{a}{b}-\frac{b}{a}}, \text{ or, } y^{\frac{a^2-b^2}{a\,b}} = \frac{b\, d^{\frac{1}{b}}\, a^{\frac{a}{b}}}{a\, c^{\frac{1}{a}}\, b^{\frac{b}{a}}};$$

$$\text{whence } y = \left\{ \frac{b\, d^{\frac{1}{b}}\, a^{\frac{a}{b}}}{a\, c^{\frac{1}{a}}\, b^{\frac{b}{a}}} \right\}^{\frac{a\,b}{a^2-b^2}}.$$

Proceeding in exactly the same manner, for x we obtain

$$x = \left\{ \frac{b\, c^{\frac{1}{b}}\, a^{\frac{a}{b}}}{a\, d^{\frac{1}{a}}\, b^{\frac{b}{a}}} \right\}^{\frac{a\,b}{a^2-b^2}}.$$

4. (1) If a and b be the two given numbers, m, the number of means, d the com. diff.

then (Art. 113, I.), $b = a+(m+1)\, d \ \therefore d = \dfrac{b-d}{m+1}$;

thus d is found, and the required terms are—

$$a+d,\ a+2\, d,\ a+3\, d \ . \ . \ . \ . \ . \ a+m\, d.$$

(2) Since the progression is equidifferent (the difference of every two successive terms being the same), it is plain that whatever number of means is inserted between the first and second terms, the same number may be inserted between every successive pair, and thus a new series in arithmetical progression be formed. Also, as every pair of terms may be considered as the first and last of a separate progression, the same number of arithmetical means may be inserted between every two terms, as between the first two. By similar reasoning, it may be shown that an analogous proposition exists for geometrical progression.

5. (1) Art. 165 (I).

(2) Art. 165, $P_2 = m \,.\, m-1$;

which in the present case (as the vowel is always the central letter) becomes

$$P_2 = 8 \times 7 = 56.$$

6. Art. 231.
7. (1) Art. 238. (2) Art. 237.
8. (1) Art. 288.

(2) Let ϕ be the given angle, and the equation to the given line

$$y = m\,x + b;$$

then the equation to the line required, passing through the point (x, y), will be (291) of the form

$$y - y' = m'(x - x').$$

The tangent (ϕ) of the angle of the two intersecting lines is found by taking the difference of the tangents of the two lines, m and m':

$$\therefore \tan\phi = \pm \frac{m-m'}{1+m\,m'} \dots\dots\dots\dots (a)$$

$$\text{But } m' = \frac{m - \tan\phi}{1 + m\tan\phi} \text{ or } \frac{m-\tan\phi}{1-m\tan\phi}.$$

Substituting, we obtain two equations:

$$y - y' = \frac{m-\tan\phi}{1+\tan\varphi}(x-x');$$

$$y - y' = \frac{m+\tan\phi}{1-m\tan\phi}(x-x').$$

The two equations represent the two lines which can generally be drawn through the same point, making a given angle with a right line.

In equation (a) the lines are *parallel* if $m = m'$; and *perpendicular* to each other if $1 + m\,m' = 0$.

(3) Art. 308.

1857. *October 28th.*

1. Ex. 9, p. 57.
2. (1) Arts. 41 and 46. (2) G.C.M. is $x+$
3. (1) $x=1$. (2) $x=3, y=5$.
   (3) Let x and $(60-x)$ be the parts,

Then $x\ (60-x) : x^2+(60-x)^2 :: 2 : 5$ by the question;

$$\text{whence } 5\,x\,(60-x) = 2\,x^2+2\,(60-x)^2.$$

$$\therefore x^2-60\,x=-800;\ x=30\pm\sqrt{900-800};\ =30\pm10;\ =40 \text{ and } 20$$

4. (1) Since $a : b :: c : d$. $\therefore \frac{a}{b}=\frac{c}{d}$.

Multiplying both fractions by $\frac{m}{n}$, we have $\frac{m\,a}{n\,b}=\frac{m\,c}{n\,d}$;

$$\text{or } m\,a : n\,b :: m\,c : n\,d;$$

$$\text{whence } m\,a\pm n\,b : m\,c\pm n\,d :: m\,a : m\,c;$$
$$:: a : c.$$

Again, since $\frac{a}{b}=\frac{c}{d}$, multiplying both fractions by $\frac{p}{q}$, we have $\frac{p\,a}{q\,b} = \frac{p\,c}{q\,d}$; whence, as above,

$$p\,a\pm q\,b : p\,c\pm q\,d :: a : c.$$

$$\text{But } m\,a\pm n\,b : m\,c\pm n\,d :: a : c;$$

$$\therefore m\,a\pm n\,b : p\,a\pm q\,b :: m\,c\pm n\,d : p\,c\pm q\,d.$$

(2) Art. 146.

5. (1) Arts. 211, 212. (2) Art. 217.
6. Art. 241.
7. (1) Art. 283, III.

(2) Art. 289. From the equation, $x=-\frac{3}{2}$, $y=3$. Substitute these values for x and y in the formula.

(3) For the length of the perpendicular, we have (Art. 295)—

$$p=\frac{y'-m\,x'-b}{\sqrt{m^2+1}}.$$

From the given equation, we have $y'=0$, $m=2$, $x'=0$, $b=3$.

$$\therefore p=\frac{-3}{\sqrt{2^2+1}}=-\frac{3}{\sqrt{5}}.$$

Verification.—See fig. XI. or take a right-angled triangle and mark the right angle A, and the other angles B and L. Make $A\,B=3$, $A\,L=-\frac{3}{2}$; and draw $A\,P$ the required perpendicular,

then the triangle $A B L = \frac{1}{2} A B \,.\, A L = \frac{1}{2} A P \,.\, B L$; and $B L = \sqrt{A B^2 + A L^2}$. But $A P \,.\, B L = A B \,.\, A L$,

$$\therefore A P = \frac{A B \,.\, A L}{B L} = \frac{3\left(-\frac{3}{2}\right)}{\sqrt{(3)^2 + \left(-\frac{3}{2}\right)^2}}$$

$$= -\frac{9}{\sqrt{45}} = -\frac{9}{3\sqrt{5}} = -\frac{3}{\sqrt{5}}.$$

(4) Art. 342.

1858. *October* 28*th*.

1. Let x = No. of miles per hour, or rate of the train's motion.
y = No. of hours of performing the journey.
$\therefore x\,y$ is the length of the line.

Then No. of miles run before the accident will be x, and since, including the delay, the whole time of performing the journey was $y+3$ hours, the time of actual travelling was $y+2$ hours; of which one elapsed before the accident. Consequently, the distance from the scene of the accident to the destination of the train, is represented by $\frac{3\,x}{5}(y+1)$, and

$$x + \frac{3\,x}{5}(y+1) = x\,y, \text{ or } 5\,x + 3\,x\,y + 3\,x = 5\,x\,y.$$

$$\therefore 8\,x = 2\,x\,y, \text{ and } y = 4.$$

Moreover, the difference mentioned in the second part of the question arises entirely from the different rate at which the 50 miles were passed over; the time of passing 50 miles at x miles per hour, is $\frac{50}{x}$, and at $\frac{3}{5}$ of x miles per hour $\frac{50}{\frac{3\,x}{5}}$ or $\frac{250}{3\,x}$.

$$\therefore \frac{250}{3\,x} - \frac{50}{x} = 1\tfrac{1}{3}.$$

$$(\times 3\,x) \qquad 250 - 150 = 4\,x.$$

$$x = 25.$$

$$\therefore x\,y = 100 \text{ miles, the length of the line.}$$

2. (1) $a\,b\,c + (a\,b + a\,c + b\,c)\,x + (a+b+c)\,x^2 + x^3$.
(2) $x+y)\,(x^2 - x\,y + y^2)\,(x-y)\,(x^2 + x\,y + y^2)$.
(3) (4) Art. 82, α and β.

3. (1) Art. 189. (2) Arts. 191, 193. (3) Art. 192.

(4) $2^3 \times 3 = 24$ $\therefore \log \overline{2 \times 3} + \log 3 = \log 24 = 1{\cdot}380211$.

(5) $3^3 \times 2 = 54$ $\therefore \log \overline{3 \times 3} + \log 2 = \log 54 = 1{\cdot}732393$.

(6) $\log 3 \times \log 2 = \log 6 = {\cdot}778151$ $\therefore \log 5{\cdot}4 = 0{\cdot}732393$.

$\therefore \log {\cdot}006 = \overline{3}{\cdot}778151$.

4. (α) $x = 10$. (β) $x = 236, y = -145$.

$$x^2 + x\,y + y^2 = 7 \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$x - y = 3 \quad \ldots\ldots\ldots\ldots \quad (2)$$

$(1)-(2)^2$
$$3\,x\,y = -2$$

$$x\,y = -\tfrac{2}{3} \quad \ldots\ldots\ldots\ldots \quad (3)$$

$(1)+(3)$
$$(x+y)^2 = 6\tfrac{1}{3}$$

$$x + y = \pm\sqrt{\tfrac{19}{3}} \quad \ldots\ldots\ldots \quad (4)$$

$\frac{(1)+(4)}{2}$
$$x = \frac{3 \pm \sqrt{\frac{19}{3}}}{2}$$

$\frac{(4)-(1)}{2}$
$$y = \frac{\pm\sqrt{\frac{19}{3}} - 3}{2}.$$

(1) Arts. 176 and 181. (2) Art. 184.

5. Arts. 231 and 232.

6. Art. 237.

7. (1) Art. 288. (2) Art. 289. (3) Art. 299.

1859. *July 19th.*

1. Let x and y be the number of lbs. of black and green tea respectively, and let the cost price be a and b shillings per pound.

$\therefore$ cost price of black is $a\,x$, and of green $b\,y$,

$\therefore$ cost price of mixture is $a\,x + b\,y$,

and selling price $(a\,x + b\,y)\,1{\cdot}04$.

Then by question,

$$(a\,x + b\,y)\,1{\cdot}04 = 1{\cdot}05\,a\,x + 1{\cdot}03\,b\,y.$$

$$\therefore 104\,a\,x + 104\,by = 105\,a\,x + 103\,by.$$

$$\therefore a\,x = b\,y,$$

$$\frac{x}{y} = \frac{b}{a},$$

$$\text{or } x : y :: b : a.$$

That is, the proportion of each must be inversely as the prices per lb.

2. (1) $\dfrac{\frac{2}{3}+\frac{4}{7}}{\frac{2}{3}-\frac{4}{7}}=\dfrac{\frac{26}{21}}{\frac{2}{21}}=13;\ \therefore \dfrac{1}{13}\,\dfrac{\frac{2}{3}+\frac{4}{7}}{\frac{2}{3}-\frac{4}{7}}=1.$

$$2\left(\frac{3}{8}-\frac{1}{12}\right)=\frac{7}{24}\times 2=\frac{7}{12};$$

$$\therefore \frac{3}{4}+\frac{5}{6}-2\left(\frac{3}{8}-\frac{1}{12}\right)=\frac{9+10-7}{12}=1;$$

and the whole expression $= 0$.

(2) $2\sqrt{1+x}$. (3) 0.

3. (1) Art. 40.

(2) H will generally be the G.C.M. of P and Q, when numerical values are given to the letters, but not always. $a+b$ is generally the G.C.M. of a^2-b^2, and $a^2+2\,a\,b+b^2$: but in some cases it is not;—*e.g.* if a and b are both odd numbers or both even; or if a is a multiple of b; &c.

4. (1) and (2) Art. 146.

(3) If $a:b::c:d$, $d=\dfrac{b\,c}{a}$. Substitute this in the given expression, and it becomes

$$\frac{b^3\,c^3}{a^2}+b\,c^3 : a^2\,b\,c+b^3\,c :: c^2\,b^4+\frac{b^4\,c^4}{a^2} : a^2\,b^4+b^4\,c^2;$$

$$\text{or } \frac{b^2\,c^2}{a^2}+c^2 : a^2+b^2 :: c^2+\frac{c^4}{a^2} : a^2+c^2.$$

Multiplying extremes and means gives identical results.

(4) If b and d are very nearly equal, so are a and c; and if we substitute one for the other, say b for d, and a for c,

$$a\,b^3+b\,a^3 : a^3\,b+b^3\,a :: a^2\,b^4+a^2\,b^4 : a^2\,b^4+a^2\,b^4,$$

we obtain, as a result nearly true, a proportion of equality such as is also $3\,d-2\,b$, or d (very nearly) : d.

5. (1) Art. 166. (2).

6. (1) Art. 116.

(2) No. For if possible, Let x be the first of three numbers, both in arithmetical and geometrical progression,

$d =$ common difference,
$r =$ common ratio.

Then x, $(x+d)$, $(x+2\,d)$, are in arithmetical progression.

x, $r\,x$, $r^2\,x$, are in geometrical progression.

Also since the three numbers are the same in both progressions, we have—

$$r\,x = (x+d) \ldots\ldots\ldots\ldots (1)$$
$$r^2\,x = (x+2\,d) \ldots\ldots\ldots\ldots (2)$$
$$\text{From (1), } (r-1)\,x = d \ldots\ldots\ldots\ldots (3)$$
$$(2), (r^2-1)\,x = 2\,d \ldots\ldots\ldots\ldots (4)$$

Divide (4) by (3),

$$\therefore r+1 = 2,$$

$$\text{and } r = 1.$$

$$\text{But } r\,x = x+d, \text{ from (1)},$$

$$\text{or } x = x+d; \text{ since } r = 1.$$

$$\therefore d = 0.$$

But x, $(x+0)$, $(x+2\times 0)$ are *not* in arithmetical progression (112) whatever x may be.

(3) $s =$ sum of $a+a\,r+a\,r^2+a\,r^3+a\,r^4+\;\ldots\ldots+l$

$$s' = \ldots\ldots \frac{1}{a}+\frac{1}{a\,r}+\frac{1}{a\,r^2}+\ldots\ldots\ldots\ldots+\frac{1}{l}$$

$$s = a\,(1+r+r^2+r^3+r^4+\ldots\ldots\ldots+r^{n-1}) = a\,\frac{1-r^n}{1-r},$$

$$\therefore s' = \frac{1}{a}\left(1+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\;.\;+\frac{1}{r^{n-1}}\right) = \frac{1}{a}\left(\frac{1-\dfrac{1}{r^n}}{1-\dfrac{1}{r}}\right).$$

$$\frac{s}{s'} = \frac{a\,.\,\dfrac{1-r^n}{1-r}}{\dfrac{1}{a}\,.\,\dfrac{1-1\frac{1}{r^n}}{1-\frac{1}{r}}} = a^2\,\frac{r^{n-1}-r^n}{1-r} = a^2\,r^{n-1};$$

$$\text{but } a\,r^{n-1} = l \quad \therefore \frac{s}{s'} = a\,l.$$

7. (1) Conclusion of Art. 167.

(2) Let $I =$ the income, $m =$ the given number of years, and suppose the part added to the capital increases the income by an (x)th part; then we have

Income at the end of the		1st year		$(1+x)\,.\,I.$
"	"	2nd	"	$(1+x)^2\,.\,I.$
"	"	3rd	"	$(1+x)^3\,.\,I.$
"	"	*m*th	"	$(1+x)^m\,.\,I.$

But by the question,

$$(1+x)^m\,.\,\mathrm{I} = n\,.\,I.$$

$$\therefore (1+x)^m = n,$$

$$x = n^{\frac{1}{m}}-1.$$

8. (1) $\frac{x+6}{x-1}+\frac{x-6}{x+1}=\frac{2\,(x^2-6)}{x^2-1}$

$$\frac{x+6}{x-1}+\frac{x-6}{x+1}=\frac{x^2+7\,x+6+x^2-7\,x+6}{x^2-1};$$

$$\therefore \frac{2\,(x^2+6)}{x^2-1}=\frac{2\,(x^2-6)}{x^2-1};$$

$$\therefore x^2+6=x^2-6$$

$$x^2-x^2=12,$$

$$(x+x)\,(x-x)=12;$$

$$\text{but } x-x=0,$$

$$x+x \text{ or } 2\,x=\frac{12}{0}=\propto.$$

(2) $$4\,x^2+7\,x\,y+2\,y^2=13 \quad \ldots\ldots\ldots\ldots (1)$$

$$5\,x\,y+7\,y^2=12 \quad \ldots\ldots\ldots\ldots (2)$$

(+) $$4\,x^2+12\,x\,y+9\,y^2=25;$$

$$\therefore 2\,x+3\,y=\pm 5 \quad \ldots\ldots\ldots\ldots (3)$$

$$y=\frac{5-2\,x}{3};$$

Substitute in (2)

$$\frac{25\,x-10\,x^2}{3}+\frac{175-140\,x+28\,x^2}{9}=12,$$

$$75\,x-30\,x^2+175-140\,x+28\,x^2=108,$$

$$2\,x^2+65\,x-67=0,$$

$$x=\frac{-65\pm\sqrt{4025+536}}{4}=\frac{-65\pm 69}{4},$$

$$x=1 \text{ or } -\frac{67}{2},$$

$$y=1 \text{ or } 24.$$

9. (1) Art. 191

(2) First $7^1=7$, $7^2=49$, $7^3=343$, and since 234 lies between 49 and 343, it is plain that the characteristic of 234 to a base 7 is 2. Similarly the characteristic of ·0067 may be shown to be $\bar{2}$.

(3) Art. 192.

(4) First $\frac{9\cdot9329}{9932\cdot8}=\cdot001$ *very* nearly,

and log $9\cdot9329-\log 9932\cdot8=\log\cdot001$, or $\log\cdot001=\bar{3}\cdot0000044$.

Let the value of $\cdot\overline{001}|^{\cdot 001} = x$;

$$\therefore \cdot 001 \log \cdot\overline{001}| = \log x;$$

$$\text{or } \cdot 001 \times \bar{3}\cdot 0000044 = \log x;$$

$$\text{or } \frac{\bar{3}\cdot 0000044}{1000} = \log x.$$

$$\therefore \log x = \bar{1}\cdot 9970000 \text{ and } x = \cdot\overline{001}|^{\cdot 001} = 0\cdot 9931160$$

10. (1) Art. 321. (2) Art. 322. (3) Art. 325.
11. (1) Art. 199. (2) Art. 212. (3) Art. 228.
    [1] 36°. [2] 27° 75′. [3] 108°.
12. (1) Art. 233. (2) Art. 219. The tan $90° = \infty$.
13. (2) Art. 234. (2) p. 149, area = 600 square feet.
14. (1) Art. 241. (2) Art. 242.

    (3) First $B = 180° - (A + C) = 39°\ 14'\ 25''$;

$$\text{But } \frac{c}{a} = \frac{\sin C}{\sin A} = \frac{\sin 88°\ 30'}{\sin 52°\ 15'\ 35''};$$

$$\therefore c = \frac{500 \sin 88°\ 30'}{\sin 52°\ 15'\ 35''} = 632\cdot 06.$$

(4) In logarithmic tables the radius is assumed equal to ten thousand millions, in order to avoid the introduction of negative indices.

Thus, since to the radius 1 we have $\sin 1' = \cdot 000290882$, $\therefore$ to radius 10^{10}, we shall have $\sin 1' = 2908882$, and hence $\log \sin 1' = \log 2908882 = 6\cdot 4637261$.

1859. *October* 26*th*.

1. 1 ton of ore = 2240 lbs. = 15680000 grs.

$$\therefore 800 : 6 :: 15680000 : x = \frac{6 \times 156800}{8} = \frac{3 \times 156800}{4}$$

= 117600 grs. = 4900 dwts. = 245 oz., which, at 5*s*. per oz., are worth £61 5*s*.

2. (1) G.C.M. of the fraction is $x+3$: dividing both terms by this quantity, gives $\dfrac{2x^2 - 6x + 5}{3x^2 - 5}$.

$$(2)\ \frac{x^2 - xy + y^2}{x^2 y^2}\left\{\frac{1}{x} + \frac{1}{y}\right\} = \frac{x^3 + y^3}{x^3 y^3} = \frac{x^3 z^3 + y^3 z^3}{x^3 y^3 z^3}.$$

$$\frac{y^2 - yz + z^2}{y^2 z^2}\left\{\frac{1}{y} + \frac{1}{z}\right\} = \frac{y^3 + z^3}{y^3 z^3} = \frac{x^3 y^3 + x^3 z^3}{x^3 y^3 z^3}.$$

Subtracting the latter result from the former, we obtain,

$\frac{y^3 z^3 - x^3 y^3}{x^3 y^3 z^3}$ or $\frac{z^3 - x^3}{x^3 z^3}$, which is also obtained from the right member of the given equation.

3. (1) $x = 13$ or $-4\frac{1}{3}$.

(2) $x = \frac{a+b}{2} \pm \frac{1}{2} \sqrt{a^2+b^2+4c^2+2ab-4(a+b)c}$.

(3) Let x be the mean and y the ratio, $\therefore$ the three Nos. are $\frac{x}{y}$, x, and xy. Their product $x^3 = 729$ $\therefore x = 9$.

$$\text{Also } \frac{x^2}{y^2} + x^2 + x^2 y^2 = 819; \text{ or } 81(1+y^2+y^4) = 819 y^2.$$

Dividing by 9 and transposing $9y^4 - 82y^2 + 9 = 0$,

$$y^2 = \frac{82 \pm \sqrt{6724-324}}{18} = \frac{82 \pm 80}{18} = 9 \text{ or } \frac{1}{9}.$$

$$\therefore y = \pm 3 \text{ or } \pm \frac{1}{3}.$$

$$\left.\begin{array}{llrrr} y = & 3 & \text{gives} \quad 3, & 9, & 27 \\ y = & \frac{1}{3} & \text{,,} \quad 27, & 9, & 3 \\ y = & -3 & \text{,,} \quad -3, & 9, & -27 \\ y = & -\frac{1}{3} & \text{,,} \quad -27, & 9, & -3 \end{array}\right\} \text{all of which fulfil the conditions of the problem.}$$

4. (1) Present value $= p \dfrac{1 - \left(\frac{1}{1+r}\right)^n}{r}$, Art. 169.

(2) When the annuity is perpetual $n = \infty$ and $\left(\frac{1}{1+r}\right)^n$ vanishes, and the present value $= \frac{p}{r}$.

(3) This expression, when the interest is 5 per cent. per annum, is $\frac{1}{\cdot 05} = 20$. $\therefore$ a freehold annual rent of £p, is the same as a perpetual annuity of £p; and the present value of the freehold is 20 times the annual rent, or in the common phrase is worth 20 years' purchase.

5. (1) Art. 194.

(2) Log $16 = \log 2^4$, and $\log \frac{1}{16} = \log 1 - \log 16$
$= 0 - 1{\cdot}204120 = {\cdot}795880$.

$$\therefore \log 6{\cdot}25 = {\cdot}795880.$$

$$\log \frac{1}{16}, \text{ or } {\cdot}625 = \bar{1}{\cdot}795880.$$

$$\log {\cdot}000625 = \bar{4}{\cdot}795880.$$

(3) Art. 197 ***.

(4) 9·980538. Art. 215. Log of tan subtracted from unity (141 Note), gives log of cot.

6. (1) Arts. 231, 232.

(2) $\sin (A+B) = \sin A \sqrt{(1-\sin^2 B)} + \sin B \sqrt{(1-\sin^2 A)}$.

7. (1) Art. 243. (2) $A = 115° \; 36' \; 58{\cdot}15''$, $B = 27° \; 3' \; 1{\cdot}85''$.

8. (1) Art. 282. (2) Art. 294.

(3) The plane must cut the cone parallel to one of its sides (273).

1860. *July* 17*th*.

1. (1) Let price of consols be x per cent.,

$$\therefore \frac{3{\cdot}15 \, (x+\frac{1}{8})}{100} = 3, \text{ or } 3{\cdot}15 \, (x+\tfrac{1}{8}) = 300,$$

$$\therefore x+\tfrac{1}{8} = \frac{300}{3{\cdot}15} = 95{\cdot}\dot{2}3809\dot{5},$$

$$\therefore x = 95{\cdot}113\dot{0}9523\dot{8} = £95 \;\; 2s. \; 3d.$$

(2) $\dfrac{3000}{94\frac{7}{8}+\frac{1}{8}} = 3157{\cdot}894737.$

$\dfrac{2000}{112\frac{1}{8}+\frac{1}{8}} = 1781{\cdot}737194.$

3 per cent. on the consols is . . 94·736842, or £94 14*s*. 8¾*d*.
6 per cent. on the Canada bonds 106·904232, or £106 18*s*. 1*d*.

Total interest on £5000 = 201·641074, or £201 12*s*. 9¾*d*.
Interest per cent. = 4·0328215, or £4 0*s*. 7¾*d*.

2. (1) First $\dfrac{11\frac{3}{4}-10\frac{1}{2}}{11\frac{3}{4}+10\frac{1}{2}} = \dfrac{\frac{5}{4}}{\frac{89}{4}} = \frac{5}{89}$; $\dfrac{10\frac{3}{8}+11\frac{1}{8}}{10\frac{3}{8}-\;9\frac{1}{8}} = \dfrac{\frac{43}{2}}{\frac{5}{4}} = \frac{86}{5}$;

$\dfrac{\frac{2}{7}+\frac{3}{11}}{\frac{2}{7}-\frac{3}{11}} = \dfrac{\frac{43}{77}}{\frac{1}{77}} = 43.$ $\therefore \dfrac{5}{89} \times \dfrac{5}{86} \times 43 = \dfrac{5 \;\; 5 \;\; 1}{89 \;\; 2} = \dfrac{25}{178}.$

(2) $\dfrac{\frac{2435}{99900000}}{\frac{2435}{999900000}} = \frac{9999}{999} = 10\frac{1}{111} = 10{\cdot}\dot{0}0\dot{9}.$

(3) 3154. (4) $\cdot 22\dot{2}$.

3. (1) $a^3 + a \, b^2 \, c + b^2 \, c + c^3$.

(2) First, $2\,n+1$ is always an *odd* number, whatever positive integer n may be.

For brevity, let $2\,n+1 = m$.

$$\therefore x^{2n+1} + y^{2n+1} = x^m + y^m.$$

$$\text{Assume } z = x + y,$$

$$\therefore x = z - y, \text{ and}$$

$$x^m = z^m - m\,z^{m-1}\,y + \&c. \ldots\ldots + m\,z\,y^{m-1} - y^m,$$

y^m being negative when m is *odd*.

$$\therefore x^m + y^m = z^m - m\,z^{m-1}\,y + \&c. \ldots\ldots + m\,z\,y^{m-1},$$

when m is *odd*.

Now since every term of the right side contains z, it is divisible by z, or $x+y$. $\therefore x^m+y^m$ is also divisible by $x+y$, or $x^{2n+1}+y^{2n+1}$ is always divisible by $x+y$.

(3) First multiply both terms of the first member of the dividend by its own numerator, and we obtain,

$$\sqrt{\frac{2-x^2+2\sqrt{1-x^2}}{x^2}} = \frac{1+\sqrt{1-x^2}}{x},$$

then the second member, and we have,

$$\sqrt{\frac{2-x^2-2\sqrt{1-x^2}}{x^2}} = \frac{1+\sqrt{1-x^2}}{x},$$

and the dividend becomes $\frac{2}{x}$.

Similarly for the divisor,

$$\frac{6+2\sqrt{9-4x^2}}{4x} = \frac{3+\sqrt{9-4x^2}}{2x},$$

and its second member

$$\frac{6-2\sqrt{9-4x^2}}{4x} = \frac{3-\sqrt{9-4x^2}}{2x},$$

and the divisor becomes $\frac{\sqrt{9-4x^2}}{x}$,

$$\text{then } \frac{2}{x} \div \frac{\sqrt{9-4x^2}}{x} = \frac{2}{\sqrt{9-4x^2}}.$$

4. (1) Arts. 100, 101. (2) 146.

(3) Given $a : b :: c : d :: e : f$ to prove

$$(1)\quad a : b :: \sqrt{\frac{a^2c^2}{e^2} + \frac{a^2e^2}{c^2} + \frac{c^2e^2}{a^2}} : \sqrt{\frac{b^2d^2}{f^2} + \frac{b^2f^2}{d^2} + \frac{d^2f^2}{b^2}}.$$

$$\text{For } \frac{a}{b} = \frac{c}{d}, \text{ and } \frac{a}{c} = \frac{b}{d} = \frac{c}{e} = \frac{d}{f};$$

$$\therefore \frac{a^2}{c^2} = \frac{b^2}{d^2}, \ \&c.$$

$$\therefore \frac{a^4}{b^4} = \frac{a^2}{b^2}\ \frac{c^2}{d^2} = \frac{a^4+c^4+e^4}{b^4+d^4+f^4},$$

$$\therefore \frac{a^2}{b^2} = \frac{\dfrac{a^4+c^4+e^4}{c^2}}{\dfrac{b^4+d^4+f^4}{d^2}},$$

but $\dfrac{a^4}{c^2} = \dfrac{a^2\,c^2}{e^2}$, $\dfrac{c^4}{c^2} = \dfrac{a^2\,e^2}{c^2}$, and $\dfrac{e^4}{c^2} = \dfrac{e^2\,c^2}{a^2}$;

also $\dfrac{b^4}{d^2} = \dfrac{b^2\,d^2}{f^2}$, $\dfrac{d^4}{d^2} = \dfrac{b^2\,f^2}{d^2}$, and $\dfrac{f^4}{d^2} = \dfrac{d^2\,f^2}{b^2}$;

$$\therefore \frac{a^2}{b^2} = \frac{\dfrac{a^2\,c^2}{e^2} + \dfrac{a^2\,e^2}{c^2} = \dfrac{c^2\,e^2}{a^2}}{\dfrac{b^2\,d^2}{f^2} + \dfrac{b^2\,f^2}{d^2} + \dfrac{d^2\,f^2}{b^2}};$$

$$\therefore \frac{a}{b} = \frac{\sqrt{\dfrac{a^2\,c^2}{e^2} + \dfrac{a^2\,e^2}{c^2} + \dfrac{c^2\,e^2}{a^2}}}{\sqrt{\dfrac{b^2\,d^2}{f^2} + \dfrac{b^2\,f^2}{d^2} + \dfrac{d^2\,f^2}{b^2}}};$$

$$\therefore a : b :: \sqrt{\frac{a^2\,c^2}{e^2} + \frac{a^2\,e^2}{c^2} + \frac{c^2\,e^2}{a^2}} : \sqrt{\frac{b^2\,d^2}{f^2} + \frac{b^2\,f^2}{d^2} + \frac{d^2\,f^2}{b^2}}.$$

(2) $= a^3\,d\,f + c^3\,b\,f + e^3\,b\,d : b^3\,c\,e + d^3\,a\,e + f^3\,a\,c.$

$\because a : b :: c : d :: e : f,$

$$\therefore \frac{a}{b} = \frac{c}{d} = \frac{e}{f},$$

$$\therefore \frac{a^3\,d\,f}{b^3\,c\,e} = \frac{a^3\,b^2}{b^3\,a^2} = \frac{c^3\,b\,f}{d^3\,a\,e} = \frac{e^3\,b\,d}{f^3\,a\,c} = \frac{a}{b},$$

$$\therefore \frac{a^3\,d\,f + c^3\,b\,f + e^3\,b\,d}{b^3\,c\,e + d^3\,a\,e + f^3\,a\,c} = \frac{3\,a}{3\,b} = \frac{a}{b},$$

$\therefore$ &c. *Q.E.D.*

5. (1) Art. 165 (I). (2) Art. 166 (I).

(3) To prevent ambiguity we shall let C_r represent the number of combinations that can be formed out of n things taken r together, then

$$C_r = \frac{n\,(n-1)\,(n-2)\,\&c.\,(n-r+1)}{1\,.\,2\,.\,3\,.\,.\,.\,.\,r}.$$

Making r equal to the numbers 1, 2, 3, &c., $n-2$, $n-1$, n, in order, we shall have,

$$C_1 = n\,;\ C_2 = \frac{n\,(n-1)}{1\,.\,2}\,;\ \&c.,\ C_{n-2} = \frac{n\,(n-1)}{1\,.\,2},$$

$$C_{n-1} = n,\ C_n = 1.$$

Now when $r = n$, we have,

$$C_r = 1.$$

$$\therefore C_r + C_{r-1} = (n+1)_r.$$

For the second part of this question, see any demonstration of the Binomial Theorem.

The 'general reasoning' by which that theorem is established, is precisely of the same character as that which is required to show that,

$$(m+n)_r = (m)_r + (m)_{r-1}\,(n)_1 + \&c. \ldots + (n)_r,$$

but it is much too long to introduce here.

6. (1) Art. 117.

(2) First the nth term $\times r^n = (2\,n)$th term,

and the $(2\,n)$th term $\times r^n = (3\,n)$th term.

Let $x =$ the first term, $r =$ the ratio,

Then $x \times r^{n-1} = n$th term,

$x \times r^{n-1} \times r^n = (2\,n)$th term $= x\,r^{2\,n-1}$.

$x \times r^{n-1} \times r^n \times r^n = (3\,n)$th term $= x\,r^{3\,n-1}$.

Let the sum of the nth and $(2\,n)$th $= s$,

" " $(2\,n)$th and $(3\,n)$th $= s'$,

$$\text{Then } x\,r^{n-1} + x\,r^{2\,n-1} = s \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$x\,r^{2\,n-1} + x\,r^{3\,n-1} = s' \quad \ldots\ldots\ldots\ldots \quad (2)$$

$$\text{From (1) } x = \frac{s}{r^{n-1} + r^{2\,n-1}} \quad \ldots\ldots\ldots\ldots \quad (\alpha)$$

$$(2)\ x = \frac{s'}{r^{2\,n-1} + r^{3\,n-1}} \quad \ldots\ldots\ldots\ldots \quad (\beta)$$

$$\therefore \frac{s}{r^{n-1} + r^{2\,n-1}} = \frac{s'}{r^{2\,n-1} + r^{3\,n-1}};$$

$$\text{whence } s\,(r^n + r^{2\,n}) = s'\,(1 + r^n),$$

$$\text{and } r^{2\,n} - \frac{s' - s}{s}\,r^n = \frac{s'}{s},$$

an equation of the quadratic form, from which r^n and thence r may be readily found. Having found r, compute x from (α) or (β).

(3) Let a denote the first term, ρ the common ratio, l the rth term, and s the sum of n terms; then

$$l = a\,\rho^{r-1}.$$

$$s = \frac{\rho\,l - a}{\rho - 1}.$$

In these equations, substitute the given value of l and then compute s.

7. (1) Art. 159.

(2) Let R represent $1+r$, or £1, together with its interest for a year,

$n =$ the number of years,
$M =$ the amount in n years,
$D =$ the discount,
$P =$ the present value; then

$$D = \frac{M(R^n - 1)}{R^n}, \text{ and}$$

$$P = \frac{M}{R^n},$$

when compound interest is allowed.

(3) The last payment to be made n years, hence $= a\left(\frac{1}{1+r}\right)^n$

$$\text{last but one} = 3a\left(\frac{1}{1+r}\right)^{n-}$$

$$\text{last but two} = 3^2 a\left(\frac{1}{1+r}\right)^{n-2}$$

First $\qquad 3^{n\ 1} a\,\frac{1}{1+r}.$

∴ The sum to be invested =

$$3^{n-1}.\,a\frac{1}{1+r} + 3^{n-2}.\,a\left(\frac{1}{1+r}\right)^2 + \ldots + 3^2 a\left(\frac{1}{1+r}\right)^{n\ 2}$$
$$+ 3a\left(\frac{1}{1+r}\right)^{n-1} + a\left(\frac{1}{1+r}\right)^n$$

8. Here $6x^2 - (17a - 4b)x = 7ab + 10b^2 - 12a^2$

$$x^2 - \frac{17a - 4b}{6}x = \frac{7ab + 10b^2 - 12a^2}{6},$$

$$x = \frac{17a - 4b}{12} \pm \sqrt{\left(\frac{17a - 4b}{12}\right)^2 + \frac{7ab + 10b^2 - 12a^2}{6}},$$

$$x = \frac{17a - 4b}{12} \pm \frac{1}{12}(a + 16b).$$

$$\therefore x = \frac{3}{2}a + b \text{ and } \frac{4a - 5b}{3}.$$

(2) Let $x =$ the side of the solid square,
then $x^2 =$ the number of men.

Also $x + 75 =$ the side of the hollow square.

∴ the number of men in the second formation will be

$$\{2(x+75)+2(x+73)\}+\{2(x+73)+2(x+71)\}$$
$$+\{2(x+71)+2(x+69)\}.$$

Equating we have $x^2 = 12x+864$.

$\therefore x = 36$, and the number of men is 1,296.

9. (1) Art. 189. (2) Art. 195. (3) Art. 193.
(4) Art. 195.
(5) $\cdot 0000432 = 2^4 \times 3^3 \div 10^7$. See Arts. 195 and 197.

10. (1) Art. 211. (2) Art. 220. (3) Art. 223.
(4) $2n\pi+30°$, n being any integer.
(5) $\text{Sin}\ \theta = \frac{7}{8} \pm \frac{1}{8}\sqrt{-63}$.

11. (1) Art. 235.
(2) Because it is not adapted to logarithmic computation.
(3) Art. 250.

12. (1) Art. 232.
(2) $\text{Sin}\ 2A = 2 \sin A \cos A$, and $\cos 2A = 2\cos^2 A-1$.

Substitute these values in the given equation and reduce, wher there will result, $\cos A = \cos A$, an identical equation.

(3) $\text{Cot}\ 2A = \frac{\cot A}{2} - \frac{1}{2\cot A}$.

13. (1) Apply art. 294. (2) An angle of 90° (295).

14. (1) Art. 300.
(2) $x^2+y^2 = 0$, indicates the origin of rectangular co-ordinates.

$$x^2-y^2=0;$$

$$\text{here } x=y \text{ and } -x=-y,$$

which equations indicate a straight line, forming an angle of 45° with the axes.

(3) In fig. XVII., Art. 299, suppose C to be the origin of co-ordinates, then the sum of the squares of the distances, of the point P from the three angles P, C, M, is $0^2+r^2+y^2$, which is true on whatever part of the circumference P is taken.

15. (1) Art. 308. (2) Art. 308. (3) Art. 309.

(4) In fig. XXVI., Art. 318, let TP be one of the tangents, and suppose another drawn from T to touch the lower branch of the parabola in Q, and let CA produced be the third tangent, cutting TQ in C', then it is easy to see, by similar triangles, that $SC = SC'$ where S is the focus.

1860. *July 17th.*

1. By the question, 15 men worked for 10 days, and 25 men also for 10 days, since the work was finished in 20 days.

$$\text{Now } 15 \times 10 + 25 \times 10 = 400,$$

that is, the work was equivalent to the work of 1 man for 400 days.

$$\text{But } 15 \times 21 = 315,$$

or, the work of the 15 men for 21 days was equivalent to the work of 1 man for 315 days.

$$\text{And } 400 - 315 = 85,$$

that is, the difference equals the work of 1 man for 85 days, or of 15 men for $\frac{85}{15} = \frac{17}{3} = 5\frac{2}{3}$ days, which is the time '*behind-hand*' that the 15 men would have been. In other words, they would have required $26\frac{2}{3}$ days.

Verification.

$$15 \times 26\tfrac{2}{3} = 15 \times 10 + 25 \times 10.$$

2. (1) $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$.

(2) $1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{31}{120}x^5 + \&c.$

(3) Multiplying numerator and denominator by $1 - \sqrt{3}$, we get $\frac{-1-\sqrt{3}}{-2}$, or $\frac{1+\sqrt{3}}{2}$.

$$(4) = \sqrt{\frac{\dfrac{-1+1-x^2}{1-x^2}}{\dfrac{1-1+\dfrac{1}{x^2}}{1-\dfrac{1}{x^2}}}} = \sqrt{\frac{\dfrac{-x^2}{1-x^2}}{\dfrac{\dfrac{1}{x^2}}{1-\dfrac{1}{x^2}}}}$$

$$= \sqrt{\frac{\dfrac{-x^2}{1-x^3}}{\dfrac{1}{x^2-1}}} = \sqrt{\frac{x^2-x^4}{1-x^2}} = \sqrt{x^2}$$

$= x$ the answer required.

(5) Assume $\sqrt{5+\sqrt{6}+\sqrt{10}+\sqrt{15}}=\sqrt{u}+\sqrt{v}+\sqrt{w}$, squaring

$$5+\sqrt{6}+\sqrt{10}+\sqrt{15}=u+v+w+2\sqrt{uv}+2\sqrt{uw}+2\sqrt{vw},$$

$$\therefore u+v+w=5,$$

$$\text{and } 2\sqrt{uv}=\sqrt{6},\ 2\sqrt{uw}=\sqrt{10}, \text{ and } 2\sqrt{vw}=\sqrt{15};$$

$$\text{also } 4\,u\,v=6,\ 4\,u\,w=10, \text{ and } 4\,v\,w=15.$$

Multiplying these last equations into each other,

$$64\,u^2\,v^2\,w^2=900$$

$$\therefore 8\,u\,v\,w=30$$

$$u\,v\,w=\frac{15}{4}$$

$$\therefore u=\frac{u\,v\,w}{v\,w}=1$$

$$v=\frac{u\,v\,w}{u\,w}=\frac{3}{2}$$

$$w=\frac{u\,v\,w}{u\,v}=\frac{5}{2}$$

and the square root $=1+\sqrt{\frac{3}{2}}+\sqrt{\frac{5}{2}}=3\cdot805883.$

3. (1) $x=-\frac{b}{2a}\pm\frac{1}{2a}\sqrt{b^2-4ac}.$

(2) Art. 182, p. 108, $x^2-(\alpha+\beta)\,x+\alpha\,\beta=0.$

(3) First $\frac{1}{\alpha^3}+\frac{1}{\beta^3}=\frac{b\,(3\,a\,c-b^2)}{c^3}.$

From the given equation we have,

$$x^2+\frac{b}{a}x+\frac{c}{a}=0,$$

and from the theory of equations (conclusion of Art. 182),

$$\alpha+\beta=-\frac{b}{a}, \text{ and } \alpha\,\beta=\frac{c}{a}.$$

Cubing both equations,

$$\alpha^3+3\,\alpha^2\,\beta+3\,\alpha\,\beta^2+\beta^3=-\frac{b^3}{a^3}\ \ldots\ldots\ (1)$$

$$\alpha^3\,\beta^3=\frac{c^3}{a^3}\ \ldots\ldots\ldots\ldots\ (2)$$

Divide (1) by (2),

$$\therefore \frac{1}{\beta^3}+\frac{3}{\alpha\,\beta^2}+\frac{3}{\alpha^2\,\beta}+\frac{1}{\alpha^3}=-\frac{b^3}{c^3},$$

$$\text{and } \frac{1}{\alpha^3}+\frac{1}{\beta^3}=-\frac{b^3}{c^3}-\frac{3}{\alpha\,\beta}\left(\frac{1}{\beta}+\frac{1}{\alpha}\right)$$

$$=-\frac{b^3}{c^3}-\frac{3}{\alpha\,\beta}\left(\frac{\alpha+\beta}{\alpha\,\beta}\right).$$

$$\text{But } \alpha+\beta=-\frac{b}{a}, \text{ and } \alpha\,\beta=\frac{c}{a},$$

$$\therefore \frac{1}{\alpha^3}+\frac{1}{\beta^3}=-\frac{b^3}{c^3}-\frac{3\,a}{c}\left(-\frac{b}{c}\right)$$

$$=-\frac{b^3}{c^3}+\frac{3\,a\,b}{c^2}$$

$$=-\frac{b^3}{c^3}+\frac{3\,a\,b\,c}{c^3}$$

$$=\frac{b\,(3\,a\,c-b^2)}{c^3}.$$

(4) Multiplying (1) by 3, and (2) by 4, and adding,

$$\therefore 10\,x^2+x\,y=42.$$

$$\text{Whence } y=\frac{42-10\,x^2}{x} \quad \ldots\ldots\ldots \quad (3)$$

Substitute this value of y in (2),

$$\therefore x^2-84+20\,x^2+\frac{5292-2520\,x^2+300\,x^4}{x^2}=3,$$

$$\text{and } 321\,x^4-2607\,x^2=-5292,$$

$$\text{whence } x^4-\frac{869}{107}\,x^2=-\frac{1764}{107}.$$

A quadratic, solving, $x^2=4$,

$$\therefore x=2,$$

whence from (3) we have $y=1$.

4. Art. 189. (2) Art. 195, 4°.

(3) The modulus is a constant multiplier between two systems, depending entirely upon the radical number of the system.

In Napier's system the modulus $= 1 \div \log_e a$, which factor converts Napier's logs into those whose base is a, and is the modulus of the system whose base is a.

In Briggs's or the common system the modulus is $= 0{\cdot}4342945$ nearly. Art 193.

(4) See explanation attached to all tables of logarithms.

5. Art. 208.

(2) Art. 223. $\tan\alpha=\tan(2\,\pi+\alpha)=\tan(4\,\pi+\alpha)=$ &c.
$=\tan(2\,n\,\pi+\alpha)$.

(3) First, for reference—

$$A = \frac{A+B}{2} + \frac{A-B}{2} = m+n,$$

$$B = \frac{A+B}{2} - \frac{A-B}{2} = m-n,$$

Let $\sin A = \sin \overline{m+n}$; $\sin B = \sin \overline{m-n}$.

$\cos A = \cos \overline{m+n}$; $\cos B = \sin \overline{m-n}$.

Then from Art. 231,

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos B - \cos A = 2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}.$$

By (1), p. 142,

$$\sin C = 2 \sin \frac{C}{2} \cos \frac{C}{2}, \text{ \&c.} = \sin \{\pi - (A+B)\}.$$

By substitution in the left member of given equation,

$$\frac{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} - 2 \sin \left\{\frac{\pi-(A+B)}{2}\right\} \cos \left\{\frac{\pi-(A+B)}{2}\right\}}{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} + 2 \sin \left\{\frac{\pi-(A+B)}{2}\right\} \cos \left\{\frac{\pi-(A+B)}{2}\right\}}$$

$$= \frac{\sin \frac{A+B}{2} \cos \frac{A-B}{2} - \cos \frac{A+B}{2} \sin \frac{A+B}{2}}{\sin \frac{A+B}{2} \cos \frac{A-B}{2} + \cos \frac{A+B}{2} \sin \frac{A+B}{2}}.$$

$$\therefore \frac{\cos \frac{A-B}{2} - \cos \frac{A+B}{2}}{\cos \frac{A-B}{2} + \cos \frac{A+B}{2}}$$

$$= \frac{2 \sin \frac{A}{2} \sin \frac{B}{2}}{2 \cos \frac{A}{2} \cos \frac{B}{2}}$$

$$= \tan \frac{A}{2} \tan \frac{B}{2}.$$

6. (1) Art. 234, p. 147.

(2) $\dfrac{\tan\frac{A}{2}}{\tan\frac{B}{2}} = \dfrac{s-b}{s-a}.$

First from Art. 238,

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

$$\tan\frac{B}{2} = \frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}},$$

$$\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)} \times \frac{s(s-b)}{(s-a)(s-c)}}$$

$$= \sqrt{\frac{(s-b)^2}{(s-a)^2}} = \frac{s-b}{s-a}.$$

(3) Art. 269.

7. Ex. 3, p. 195.

(2) By (307) the equation to this system of circles is the equation to a parabola.

8. (1) Ex. 4, p. 228.

(2) Take the simplest case;—in which the tan is at the extremity of the minor axis, and the focal distances are equal. It is then manifest that the external bisector forms one side of a rhombus, and that one of the focal distances bisects it into two equilateral triangles.

9. (1) and (2) Art. 308. (3) Art. 309.

(4) See First B.A. Solutions of this date, the preceding paper, Qu. 15 (4), p. 36.

FIRST B.A. AND FIRST B.Sc. PASS EXAMINATION.

1861. *July* 16*th*.

1. (1) Let r = the rate per cent., then $210s. \times \frac{r}{100} = 21s.$ by the question, whence $r = 10$ per cent., the rate of interest required.

(2) $6\frac{4}{81}$ days.

2. (1) Art. 89. (2) $\frac{2\frac{1}{2}+1\frac{1}{3}}{2\frac{1}{2}-1\frac{1}{3}} \times \frac{1\frac{7}{12}-1}{\frac{1}{4}+\frac{5}{3}} = 1.$

(3) $\cdot000741 \div 2\cdot47 = \cdot0003.$

3. (1) $\sqrt{x^4-4x^3+2x^2+4x+1} = x^2-2x-1.$

(2) Adding we shall have

$$\frac{1-x-\sqrt{2x+x^2}+1-x+\sqrt{2x+x^2}}{\sqrt{(1-x)^2-(2x+x^2)}} = \frac{2(1-x)}{\sqrt{1-4x}}$$

4. (1) Since $\frac{a}{b} = \frac{c}{d}$, we have $a : c :: b : d$, whence $a+c : c :: b+d : d$ and $\frac{a+c}{b+d} = \frac{c}{d} = \frac{a}{b}$, by the question.

(3) Let $x = my$, then $x^2 = m^2y^2$, and $x^2+y^2 = (m^2+1)y^2$, also $x^2-y^2 = (m^2-1)y^2$. Now since m^2+1 is constant, and also m^2-1; it is plain that x^2+y^2 will vary as x^2-y^2.

5. (1) Art. 164. (2) Art. 165.

6. (1) Art. 114.

(2) In the series $2+3\frac{1}{2}+5+$&c., we have

$$a = 2,\ d = \frac{3}{2},\ l = \left\{2+\frac{3}{2}(n-1)\right\}$$

∴ III. Art. 114 becomes

$$S = \left\{2+\frac{3}{4}(n-1)\right\} n.$$

(3) In the geometric series $2+3\frac{1}{2}+6\frac{1}{8}+$&c., we have

$$a = 2 \text{ and } r = \frac{7}{4};$$

substituting these values in I. Art. 117, we have

$$S = \frac{8}{3}\left\{\left(\frac{7}{4}\right)^n-1\right\}.$$

(4) Art. 114, II.

7. (1) Ex. 8, p. 117. (2) Ex. 9, p. 117.

(3) $x = \frac{b \mp \sqrt{2a^2 - b^2}}{2}$, $y = \frac{b \pm \sqrt{2a^2 - b^2}}{2}$.

8. (1) Let x and $x+5$ be the digits, then $11x+5 = 6x+15$, whence $x = 2$ and $x+5 = 7$, $\therefore$ 27 is the number.

9. (1) Art. 189. (2) Art. 195.

(3) Art. 192. Napierian logarithms are more used than common logarithms in finding the sums of infinite series, in finding integrals, and in scientific investigations.

(4) $(3 \times 2)^2 \times 2^2 = 144$, $\therefore 2(\log 2 + \log 3) + 2 \log 2 = \log 144$, and $\cdot 0144 = 144 \div 10^4$.

(5) Art. 195.

10. (1) Art. 225. (2) $\tan = \frac{\sin}{\cos}$, also see Art. 224.

(3) $(n\,180° + A)$, n being any integer.

(4) $\tan^4\theta - \frac{10}{3}\tan^2\theta = -1$, solving we get $\tan^2\theta = 3$ and $\frac{1}{3}$, and $\tan\theta = \sqrt{3}$ and $\frac{1}{3}\sqrt{3}$.

11. (1) Art. 231.

(2) $\cot A - \tan A = \frac{\cos^2 A - \sin^2 A}{\sin A \cos A} = \frac{2\cos 2A}{\sin 2A} = 2\cot 2A$.
$\therefore \cot A - 2\cot 2A = \tan A$.

(3) Here we have

$$\frac{\sin(60° + \theta) + \cos(150° - \theta)}{\cos(60° + \theta) + \sin(150° - \theta)}.$$

Expand all the expressions, observing that $\sin 60° = \cos 30°$, $\cos 60° = \sin 30°$, $\sin 150° = \sin 30$, and $\cos 150° = -\cos 30°$. Substitute these values and reduce, and we get

$$\frac{2\sin 30° \sin\theta}{2\sin 30° \cos\theta} = \tan\theta.$$

12. (1) Art. 254.

(2) Here $AC : AB :: \sin B : \sin C :: 2 : 1$, whence $\sin B = 2\sin C$. But $B = 120° - C$. $\therefore \sin B = \sin(120° - C)$ and $\sin(120° - C) = 2\sin C$. Expand and divide both sides by $\cos C$. $\therefore (2 + \cos 120°)\tan C = \sin 120°$,

$$\text{or } \frac{3}{2} \tan C = \sin 120°, \text{ and } \tan C = \frac{2}{3} \sin 120°.$$

$$\therefore C = 30°, \text{ and } B = 90°.$$

13. (1) Arts. 278 . . 80.

(2) $x = 3y$ is the equation to a straight line.

(3) In $(x^2 - a^2)^2 (x^2 - b^2)^2 + c^4 (y^2 - a^2)^2 = 0$, make $x = 0$ and reduce.

$$\therefore y^2 = a^2 \pm \frac{a^2 b^2}{c^2} \sqrt{-1}.$$

Since impossible values of y result from this equation, the locus cannot cut the axis of y. Similarly let $y = 0$, and find the points (if any) in which it would cut the axis of x. Then assume different values of x, and find the corresponding values of y, and through the resulting points trace the locus.

(4) Art. 306. (5) Art. 281.

14. (1) In this case, Art. III. $\alpha^2 + \beta^2 = C^2$.

(2) Let a be the chord on the axis of x, b that on the axis of y, x' and y' the co-ordinates of the centre, and r the radius of the circle. Then we have $r^2 - y'^2 = \left(\frac{a}{2}\right)^2$ and $r^2 - x'^2 = \left(\frac{b}{2}\right)^2$, where x' and y' must each be less than r.

15. (1) Art. 321. (2) Art. 325.

(3) Through five points, provided no three be in a straight line, a conic section, and only one, can be made to pass.

16. (1) Here $y^2 = a^2 + x^2$. Let $x = 0$, and $y = \pm a$, therefore the curve cuts the axis of y at a distance $= a$, both above and below the origin. Let $y = 0$. $\therefore x = \pm a \sqrt{-1}$, which shows that the curve does not cut the axis of x. Assume now different values for x, and get corresponding values of y, and trace the curve.

(2) Art. 336.

(3) Let in fig. XXVI., p. 212, the area $T A C = a^2$, and $y y' = 2 m (x + x')$ be the equation to the tangent. Now $\frac{1}{2} A T A C = a^2$, and $A T$ is the value of x in the equation to the tangent, when $y = 0$, also $A C$ is the value of y when $x = 0$.

Making $x = 0$, we have $y y' = 2 m x'$, whence $y = 2 m \frac{x'}{y'}$, $= A C$; making $y = 0$ we have $2 m (x + x') = 0$, and $x = -x'$ $= A T$.

1862. *July 22nd.*

1. (1) £1,381 17*s.* 6*d.* (2) Annual income £336.

2. (1) A recurring decimal is one that cannot be reduced to the form $\frac{a}{2^p\,5^q}$. See Art. 88.

(2) Art. 89. (3) $\cdot 200\dot{1}2\dot{3} = \frac{200103}{999900}$.

(4) $\cdot 0\dot{1}\dot{2} + \cdot 00\dot{1}3\dot{2} = \frac{1110}{121} = 9\cdot\dot{1}7355371900826446280\dot{9}$.

3. (1) $4x^9 - 8x^8 + 35x^7 + 47x^6 - 15x^5 - 2x^4 + 12x^3 - x^2$.
(2) $x^3 - 3x + 2$.
(3) $(5x^2 - 11x + 12)^2 - (4x^2 - 15x + 6)^2 = (9x^2 - 26x + 18)(x^2 + 4x + 6)$.

4. (1) Art. 152. (2) Ex. 6, p. 84. (3) Ex. 7, p. 84.

5. (1) Art. 166. (2) 40320. (3) 5040. (4) 70. (5) 35.

6. (1) Art. 117. (2) Arts. 118 and 120.

(3) Here the ratio is $-\frac{a}{r^2}$, $\therefore$ to n terms,

$$S = \frac{a\,r^2 \left\{\left(-\frac{a}{r^2}\right)^n - 1\right\}}{-(a+r^2)},$$

and, to infinity, $S = \frac{a\,r^2}{r^2+a}$.

(4) Assume a numerical value for r, then the value of the rth term is at once given from the question. But $\frac{\text{2nd term}}{\text{1st term}} = \text{ratio}$. Substituting the values thus obtained, the sum ot n terms may be got from Art. 121.

7. (1) $(x+b)(x+c) + (x+c)(x+a) = (2x+a)(x+b)$. Divide every term by $(x+b)(x+c)$, transpose and reduce

$$\therefore x = \frac{a(b-c) - b\,c}{2\,c - b}.$$

(2) $x = 0$ and $y = a$.

(3) Get a value of x from each equation, equate these values, and reduce

$$\therefore y^4 - 29\,y^2 = -100, \text{ whence } y = \pm 5 \text{ and } \pm 2.$$
$$\therefore x = \pm 4 \text{ and } \pm 10.$$

(4) Subtract 25 from both sides;

$$\therefore x^4+x^2-25-4\sqrt{x^4+x^2-25} = 525.$$

$$\sqrt{x^4+x^2-25} = 25 \text{ and } -21.$$

$\therefore x^4+x^2-25 = 625$ and 441, whence $x^2 = 25$ and -26, and $x = \pm 5$ and $\pm\sqrt{-26}$.

8. (1) Let $a =$ the first digit, $b =$ the second $\therefore a^2+b^2 = 130$, and $10\,a+b = 10\,b+a+18$; whence $a = 9$, $b = 7$, and the number is 97.

9. (1) Art. 189.

(2) $(\sqrt{3})^8 = 81$ $\therefore$ 8 is the log of 81 to the base $\sqrt{3}$.

(3) $(x^{-\frac{1}{2}})^{-2} = x$ $\therefore -2$ is the log of x to the base $x^{-\frac{1}{2}}$.

(4) Art. 195.

(5) To deduce one system of logarithms from another, let N be any number and a and b two bases, then

$$\log_a N = x,\ \log_b N = y.$$

$$\therefore N = a^x = b^y;$$

$$\text{and } a^{\frac{x}{y}} = b;\ b^{\frac{y}{x}} = a.$$

$$(189)\ \log_a b = \frac{x}{y};\ \log_b a = \frac{y}{x}.$$

$$\therefore y \log_a b = x;\ x \log_b a = y = \frac{x}{\log_a b};$$

or the logarithm of a number to the base a may be found by multiplying the logarithm of the number to the base b by log b, or by $\frac{1}{\log_b a}$.

$$\therefore \log_a x = \log_b x \log_a b, \text{ or}$$

$$= \frac{\log_b x}{\log_b a}.$$

(6) $\text{Log}_{100} 32 = \dfrac{\log_{10} 32}{\log_{10} 100} = \dfrac{1{\cdot}505150}{2} = 0{\cdot}752575.$

10. (1) Art. 156.

(2) Let P denote the present worth of the sum m, due n years hence, at the rate r; then in n years P must amount to m at the given rate, where $P = \dfrac{m}{1+n\,r}$.

11. (1) Art. 212. (2) Art. 214.

(3) Art. 217. Since from the second equation

$$1+\sin x = 2\cos^2 x = 2-2\sin^2 x,$$

$$\therefore \sin^2 x + \frac{1}{2}\sin x = \frac{1}{2}, \text{ and } \sin x = \frac{1}{2} \text{ and } -1,$$

$\therefore x = 30°$ or $270°$, of which the latter only satisfies the first equation.

12. (1) Art. 231.

(2) $\operatorname{Sin} 2A \sin 2A - \sin^2 A = 2\sin A \cos A \times 2\sin A \cos A - \sin^2 A = 4\sin^2 A \cos^2 A - \sin^2 A = 4\sin^2 A(1-\sin^2 A) - \sin^2 A = 3\sin^2 A - 4\sin^4 A$. But from Art. 233, $\sin 3A \sin A = 3\sin^2 A - 4\sin^4 A$, the same as that obtained in the line above.

(3) $$\frac{1+\cos 2A}{\sin 2A} = \frac{1+\cos^2 A - \sin^2 A}{2\sin A\cos A}$$

$$= \frac{\sin^2 A + \cos^2 A + \cos^2 A - \sin^2 A}{2\sin A\cos A} = \frac{2\cos^2 A}{2\sin A\cos A}$$

$$= \frac{\cos A}{\sin A} = \cot A.$$

13. (1) Art. 243.

(2) Let $\theta =$ the included angle,

$$\therefore \frac{6 \times 8}{2} \times \sin\theta = 12 \text{ square feet},$$

$$\text{whence } \sin\theta = \cdot 5, \text{ and } \theta = 30°,$$

the required side $= 4$.

14. (1) Art. 289.

(2) p is the perpendicular upon the line from the origin, and α is the angle which the perpendicular makes with the axis of x produced in the positive direction.

(3) In $x^2 y = 0$. Let $x = 0$, $y = \frac{0}{0} =$ anything; $y = 0$, $x = \frac{0}{0} =$ anything $\therefore$ the locus is the axes.

(4) In the equation $y^2 + x^2 = 0$. Let $x = 0$, $\therefore y = 0$. Let $x = 1$, $\therefore y = \pm\sqrt{-1}$. Let $x = 2$, $\therefore y = \pm\sqrt{-4}$, and since the square of every number is essentially positive, this shows that the locus is a point in the origin, but that every other point is impossible.

(5) In $x - y = 4$, make successively $x = 0$, $x = 1$, &c., and the locus is easily constructeed.

15. (1) Art. 300, V. This equation is that to a circle. In it the co-efficients of x and y are equal, and it does not contain $x\,y$.

(2) Art. 302.

(3) Assume equations for two straight lines, and combine them with the given equations; reduce these and obtain values f r y in terms of x and the constants.

16. (1) Art. 336.

(2) For simplicity let us suppose the circle to be the conic section. Then a straight line drawn from P to the centre will be a normal to the curve, and the area of $T\,C\,T'$ will be $\frac{1}{2}\,P\,C \times T\,T'$, also the area of $P\,C\,N$ will be $\frac{1}{2}\,P\,C \times p$, where p is a perpendicular from N on $P\,C$: but it is plain that the larger $T\,T'$ is, the smaller p will be, and as $\frac{1}{2}\,P\,C$ is constant, the area of $T\,C\,T'$ will vary inversely as the area of $P\,C\,N$.

1863. *July* 21*st*.

1. (1) 2 per cent. Ex. 8, p. 88.

(2) £3 4*s*. 6*d*. per cent. Ex. 10, p. 92.

(1) Art. 83. Ex. $\cdot 237 = \frac{237}{10^3}$. (2) $\frac{3085\cdot 5}{\cdot 00051} = 6050000$.

(3) $2\cdot\dot{3}01\dot{7} = 2\frac{3014}{9990} = \frac{22994}{9990} = \frac{11497}{4995}$.

3. (1) $x^2 - 3\,x - 1$. (2) $\frac{a^4 - x^4}{a^2 - x^2}$.

(3) The first expression

$$= (x-1)\,(x^3 + 7\,x^2 - x - 1).$$

The second,

$$= (x-1)\,(x^5 + 8\,x^4 + 5\,x^3 + 5\,x^2 + 5\,x + 2).$$

4. (1) Art. 103.

(2) Since $a : b :: c : d$,

$$\therefore a+b : a :: c+d : c,$$

$$\text{and } (a+b)^3 : a^3 :: (c+d)^3 : c^3,$$

$$\text{also } -3\,a\,b^2 : b^3 :: -3\,c\,d^2 : d^3,$$

$$\therefore a^3 + 3\,a^2\,b + b^3 : a^3 + b^3 :: c^3 + 3\,c^2\,d + d^3 : c^3 + d^3$$

$$\text{whence } \frac{a^3 + 3\,a^2\,b + b^3}{c^3 + 3\,c^2\,d + d^3} = \frac{a^3 + b^3}{c^3 + d^3}.$$

(3) Suppose $a : b :: c : d$,

$$\text{then } \frac{a}{b} = \frac{c}{d} \text{ and } \frac{p\,a}{q\,b} = \frac{p\,c}{q\,d},$$

and if $p\,a$ is $> q\,b$, $p\,c$ is $> q\,d$,
if $p\,a$ is $= q\,b$, $p\,c$ is $= q\,d$,
and if $p\,a$ is $< q\,b$, $p\,c$ is $< q\,d$,

which is in accordance with Euclid's definition.

5. (1) Art. 165. (2) Ex. 5, p. 95.

6. (1) Art. 115. (2) Sum is 2.

(3) This is a geometrical progression, of which the first term is 1, and the ratio 4,

$$\therefore S = a\frac{r^n-1}{r-1} = \frac{4^n-1}{3}.$$

7. (1) $x = \pm\sqrt{-ab}$. (2) If $a = b = \pm a\sqrt{-1}$.

(3) Divide the first equation by the second,

$$\therefore x+y = \frac{a^2}{b}.$$

$$\text{But } x-y = b.$$

$$\therefore x = \frac{1}{2}\left(\frac{a^2}{b}+b\right)$$

$$\text{also } y = \frac{1}{2}\left(\frac{a^2}{b}-b\right).$$

(4) Ex. 7, p. 114.

8. (1) Suppose that A began with £x, then A's share of the gain will be $\frac{51x}{180+3x}$,

$$\text{and } x + \frac{51x}{180+3x} = 39 \text{ by the question,}$$

whence $x =$ £32·971.

9. (1) Art. 189. (2) No. (3) Art. 195. (4) $\bar{4}$, Art. 194.

10. (1) Art. 211. (2) Art. 220.

(3) Let $\theta = 0°$, $\therefore 4\theta = 0°$ and $\sin 4\theta = 0$,

also $\sin(\sin\theta) = 0$. See Art. 217.

Let $\theta = 90°$, $\therefore 4\theta = 360°$ and $\sin 4\theta = 0$.

Let $\theta = 180°$, $\therefore 4\theta = 720°$ and $\sin 4\theta = 0$, also $\sin(\sin\theta) = 0$.

(4) In (1) $\theta = \frac{180°}{7}$.

In (2) θ (in arc) $= 2·125$ nearly.

11. (1) Art. 231.

(2) From Art. 231, and formulæ on p. 40 of this Key.

$$\sin A+\sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2},$$

$$\sin A-\sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2},$$

$$\cos B-\cos A = 2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}.$$

Now to prove that

$$4 \sin 3 A \sin 5 A \sin 7 A = \sin 5 A+\sin A+\sin 9 A-\sin 15 A.$$

First, by the above formulæ,

$$\sin 5 A+\sin A = 2 \sin 3 A \cos 2 A,$$

$$\sin 9 A-\sin 15 A = -2 \sin 3 A \cos 12 A,$$

then
$$2 \sin 3 A (\cos 2 A-\cos 12 A)$$
$$= 4 \sin 3 A \sin 7 A \sin 5 A.$$

(3) For $\theta+a$ put A, and for $\theta-a$ put B; then refer to Ex. 14, p. 145.

12. (1) Art. 236.

(2) Art. 250. In this article the formulæ are applicable when the angles are small.

13. (1) Art. 306. (2) Art. 283 (III.).

(3) The equaticn to a straight line passing through a given point is (291) $y-y' = m (x-x')$. The triangle contained by the parts of the co-ordinate axes will be the area cut off, which must not exceed the greatest triangle whose base can pass through the given point.

14. (1) The equation to a circle, the co-ordinates of whose centre are a and b.

(2) The equation to a circle (301).

(3) Art. 300.

15. (1) Art. 308.

(2) Fig. XXVI., p. 212, may very easily be adapted to the following solution, by producing the normal $R P$ to meet $A Y$ in L,

and substituting K for the C given in the figure. From similar triangles, $S\,A\,K$, $S\,K\,P$, we have

(Eu. VI. 8.) $$A\,S : S\,K :: S\,K : S\,P,$$

$$\therefore S\,K^2 = A\,S\ \ S\,P,$$

also by similar triangles, $S\,A\,K$, $K\,P\,L$,

$$A\,S^2 : S\,K^2 :: K\,P^2 : K\,L^2.$$

Substitute for $S\,K^2$.

$$\therefore A\,S^2 : A\,S\,.\,S\,P :: K\,P^2 : K\,L^2,$$

$$\text{whence } A\,S : S\,P :: K\,P^2 : K\,L^2,$$

$$\text{and } A\,S : S\,P - A\,S :: K\,P^2 : K\,L^2 - K\,P^2 = L\,P^2.$$

$$\text{But } K\,P^2 : L\,P^2 :: K\,P^2 + S\,K^2 : L\,P^2 + K\,P^2,$$

$$:: S\,P^2 : K\,L^2,$$

$$\therefore A\,S : S\,P - A\,S :: S\,P^2 : K\,L^2.$$

16. (1) Art. 336.

(2) To conform to the question, the given equation must be written,

$$\frac{x}{a^2}\,X + \frac{y}{b^2}\,Y = 1,$$

and the perpendicular to this line, drawn through the point $(x'\ y')$, is found by interchanging the co-efficients of X and Y, and altering the sign of one of them;

$$\frac{x}{a^2}\,(Y - y') = \frac{y}{b^2}\,(X - x'),$$

$$\text{or } \frac{a^2\,X}{x'} - \frac{b^2\,Y}{y'} = a^2 - b^2 \quad \ldots\ldots\ldots \quad (a)$$

Suppose the ordinate $N\,P$ (Fig. XXXIII., p. 225) produced to meet the circle in Q, then the angle $Q\,C\,N = \phi$, and $C\,N = C\,Q$ cos $Q\,C\,N$, or $x' = a\cos\phi$. But $P\,N = \frac{b}{a}\,Q\,N$, and since $Q\,N = a\sin\phi$, we have $y' = b\sin\phi$.

Substituting these values of x' and y' in (a), we get

$$\frac{a\,X}{\cos\phi} - \frac{b\,Y}{\sin\phi} = a^2 - b^2.$$

(3) The intercepts cut off by the normal from the axes are obtained from (a),

$$X = \frac{a^2 - b^2}{a^2}\,x' = e^2\,x';$$

$$Y = -\frac{a^2 - b^2}{b^2}\,y' = \frac{c^2}{b^2}\,y'.$$

1864. *July* 19*th*.

1. (1) Arts. 49 and 83. (2) Art. 87. (3) ·378. (4) 4230.
(5) $\sqrt{\cdot 4} = \cdot\dot{6}$. (6) $\sqrt{10\cdot 4976} = 3\cdot 24$.

2. (1) £16038. (2) £211 15*s*. 0·66.

3. (1) Each side of the equation
$$= b\,c^2 - b^2\,c + a^2\,c - a\,c^2 + a\,b^2 - a^2\,b.$$
(2) Quotient $= x^3 - 5\,x + 3$.

(3) Result $= \dfrac{x^2+y^2}{(x+y)^2}$, which, when $x = y$, becomes $\dfrac{1}{2}$.

4. (1) Proportion is the equality of ratios, also see Art. 146.

(2) Since $a : b :: c : d$,
$$\therefore m\,a : n\,b :: m\,c : n\,d.$$
Then say 1st + 2nd : 1st − 2nd :: 3rd + 4th : 3rd − 4th,
$$\text{or } m\,a + n\,b : m\,a - n\,b :: m\,c + n\,d : m\,c - n\,d.$$
(3) Since $a : b :: c : d$,
$$\therefore (a+b) : (a-b) :: (c+d) : (c-d),$$
$$\text{and } (a+b)^4 : (a-b)^4 :: (c+d)^4 : (c-d)^4,$$
$$\text{also } 6\,a^2\,b^2 + 4\,a\,b^3 : 6\,a^2\,b^2 - 4\,a\,b^3 :: 6\,c^2\,d^2 + 4\,c\,d^3 : 6\,c^2\,d^2 - 4\,c\,d^3,$$
$$\therefore a^4 + 4\,a^3\,b + b^4 : a^4 - 4\,a^3\,b + b^4 :: c^4 + 4\,c^3\,d + d^4 : c^4 - 4\,c^3\,d + d^4.$$

5. (1) Art. 117.

(2) Here $r = \dfrac{3}{4}$, $\therefore \dfrac{a}{1-r} = \dfrac{16}{15}$.

(3) Let $x + r\,x + r^2\,x + \&c. + r^{n-1}\,x = s$,
$$x^2 + r^2\,x^2 + r^4\,x^2 + \&c. + (r^{n-1}\,x)^2 = \sigma.$$
Multiply the first equation by $1-r$, and the second by $1-r^2$,
$$\therefore x - r^n\,x = s\,(1-r), \text{ or } x\,(1-r^n) = s\,(1-r) \;.\;.\; (A)$$
$$\text{and } x^2 - r^{2n}\,x = \sigma\,(1-r^2).$$
Divide this equation by (A),
$$\therefore x + r^n\,x = \frac{\sigma}{s}\,(1+r),$$
$$\text{or } x\,(1+r^n) = \frac{\sigma}{s}\,(1+r) \;.\;.\;.\;.\;.\;.\;.\;.\;.\; (B)$$
Multiply (A) and (B) crosswise, in order to exterminate x,
$$\therefore x\,(1+r^n) \times s(1-r) = x\,(1-r^n) \times \frac{\sigma}{s}\,(1+r),$$
$$\text{whence } s^2\,(1-r)\,(1+r^n) = \sigma\,(1+r)\,(1-r^n),$$
the equation required.

6. (1) Art. 165. (2) 24. (3) 6.

7. (1) Art. 189. (2) 4. See 1862, Quest. 9. (3) Art. 195.
(4) $\mathrm{Log}_{10}\, 25 = 1{\cdot}3979400$. (5) $\mathrm{Log}_{100}\, 25 = 0{\cdot}6989700$.

8. (1) $\left(\frac{r\,t}{100} + 1\right) P$. See Art. 156.

(2) Let $p =$ present worth, and
$r =$ rate per cent., then
$p = \frac{P}{1 + n\,r}$, and making
$p = \frac{1}{2} P$, also $r = 3\frac{1}{2}$,
we get $n = \frac{2}{7}$ of a year.

9. (α) $x = \frac{5}{2}\, a$. (β) $x = 17$ and 15.
(γ) $x = -a \pm a\sqrt{5}$, also $x = -a \pm a\sqrt{37}$.
(δ) $x = 15$ and 17, $y = 17$ and 15.

10. (1) 156 is the number.

11. (1) Art. 208. (2) Art. 232.

(3) $\mathrm{Cos}\, 15° = \cos\, (45° - 30°) = \frac{1}{2\sqrt{2}}\, (\sqrt{3} + 1)$ See Arts. 227 and 228.

12. (1) Art. 234. (2) Art. 235.

13. (1) Let, in fig. XIII., p. 191, $H K$ be the given line, and $y = m\,x + b$ its equation, therefore $\tan H K F = m$; $E D$ the required line, whose equation may be assumed to be

$$y - y' = m'\,(x - x').$$

Since it passes through the point (x', y'), then $\tan E D F = m'$; also let $\tan D I K = \phi$, the given angle; then because

$$D I K = I K F - I D K, \text{ or } \phi = m - m';$$

$$\therefore \tan\phi = \pm\, \frac{m - m'}{1 + m\,m'},$$

$$\text{or } m' = \frac{m - \tan\phi}{1 + m\tan\phi} \text{ or } \frac{m + \tan\phi}{1 - m\tan\phi},$$

$$\therefore y - y = \frac{m - \tan\phi}{1 + m\tan\phi}\,(x - x') \text{ and } y - y' = \frac{m + \tan\varphi}{1 - m\tan\phi} \text{ are}$$

the required equations.

(2) The equation of the given line easily reduces to

$$y = \frac{\sqrt{3} - 1}{\sqrt{3} + 1}\, x + \frac{a}{\sqrt{3} + 1},$$

$$\therefore m = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{0{\cdot}7320508}{2{\cdot}7320508},$$

and tan $75° = t = 3{\cdot}7320508$, also $x' = 0$, $y' = 0$, the co-ordinates of the origin; substitute these values in the equation just found, and he required equations will be easily obtained.

14. (1) Art. 314.

(2) $\left.\begin{matrix} x', y' \\ x'', y'' \end{matrix}\right\}$ points in which tangents meet the curve.

Equation to tan at x', y' is $yy' = 2a(x + x')$.

But this tan passes through h, k, therefore

$$ky' = 2a\,(h + x');$$

Similarly, $ky'' = 2a\,(h + x'')$, therefore the equation to the chord of contact is $ky = 2a\,(x + h)$.

(3) Having the three lines forming the triangle, its area can be found; see 277.

15. (1) Art. 336.

(2) Equation to the normal at D, fig. XXIX.,

$$y = -a\,e^2 + \frac{x}{e}.$$

16. (1) Ex. 1, p. 235.

(2) The equation to the hyperbola (345, III.) may be put in the form,

$$y^2 = \frac{b^2\,x^2}{a^2}\left(1 - \frac{a^2}{x^2}\right)$$

$$\therefore y = \pm\frac{b\,x}{a}\sqrt{1 - \frac{a^2}{x^2}}.$$

As x increases the value of $\frac{a^2}{x^2}$ diminishes, and approaches zero. When x becomes infinite $\frac{a^2}{x^2}$ vanishes, and

$$y = \pm\frac{b}{a}\,x.$$

Now if the lines $C\,E$, $C\,E'$ (Fig. XXXVI.) make, with the axis of x, angles whose tangents equal respectively $\frac{b}{a}$, and $-\frac{b}{a}$, these lines indefinitely produced become the two asymptotes to the curve. See 346 (Def.).

1865. *Tuesday, July 18th.*

1. £3·1788.
2. (1) $2\frac{293}{1600}$. (2) ·9. (3) 85·1.

3. (1) $a^2-8ab+15b^2$. (2) $4x^2+3x+1$.

4. (1) Art. 146.

(2) If $a:b::c:d$, then $a:c::b:d$,

and (Art. 145) $a+c:a-c::b+d:b-d$,

and $a+c:b+d::a-c:b-d$;

$$\text{or } \frac{a+c}{b+d}=\frac{a-c}{b-d}\dots\dots\dots\dots (a)$$

$$\text{Similarly if } p:q::r:s \therefore \frac{p-q}{r-s}=\frac{p+q}{r+s}\dots\dots\dots\dots (b)$$

$$(a)\cdot(b) \qquad \frac{a+c}{b+d}\cdot\frac{p-q}{r-s}=\frac{a-c}{b-d}\cdot\frac{p+q}{r+s}.$$

5. (1) Art. 114 (III.).

(2) Let $x=n$, then Art. 113 (I.),

$15-3(x-1)$, or $18-3x$ is the last term.

$$\text{And (IV.)}, 45=\frac{(36-6x+\overline{3x-3})x}{2}, \text{ or } 33x-3x^2=90,$$

$$x^2-11x+30=0,$$

$$x=5 \text{ or } 6.$$

Verification. 5th term is $15-12$ or 3,

$$\text{and } S_5=\frac{(6+12)5}{2}=45.$$

6th term is $15-15$ or 0,

$$\text{and } S_6=\frac{(0+15)6}{2}=45.$$

$$(3)\ S_8=4\frac{1-\frac{1}{2}^8}{1-\frac{1}{2}}=7\tfrac{31}{32}.$$

6. (1) Art. 166. (2) 15840. See Ex. 5, p. 95.

7. (1) Art. 170.

(2) Supposing the annuity the same in both cases. The present value for n years is (Art. 169)—

$$p\,\frac{1-\left(\frac{1}{1+r}\right)^n}{r}, \text{ and for } 2n \text{ years, } p\,\frac{1-\left(\frac{1}{1+r}\right)^{2n}}{r}.$$

$$\therefore p\,\frac{1-\left(\frac{1}{1+r}\right)^n}{r}:p\,\frac{1-\left(\frac{1}{1+r}\right)^{2n}}{r}::20:24;$$

$$\text{or } 1-\left(\frac{1}{1+r}\right)^n:1-\left(\frac{1}{1+r}\right)^{2n}::5:6;$$

$$\text{or } 1 : 1+\left(\frac{1}{1+r}\right)^n :: 5 : 6.$$

$$\therefore \left(\frac{1}{1+r}\right)^n = \tfrac{1}{5}.$$

$$\therefore 1+r = \sqrt[n]{5} \text{ and } r = \sqrt[n]{5}-1.$$

Rate per cent. is $100\ \{\sqrt[n]{5}-1\}$.

8. (1) Arts. 189 and 197.

(2) $\text{Log } 10 = 1$, and $\log 5 = \log \frac{10}{2}$, or $\log 10 - \log 2$.

$$\text{Log } 25 = 2 \log 5 = 1{\cdot}397940.$$

$$\text{Log } {\cdot}00025 = \bar{4}{\cdot}397940.$$

$$\text{Log } \sqrt{{\cdot}00025} = \bar{2}{\cdot}198970.$$

9. (1) $x = 9$. (2) $x = 5$ or 6, $y = 8$ or $12\frac{3}{4}$.

(3) $$x = \frac{-2a(b+c) \pm \sqrt{5a^2(b-c)^2+4a(bc-a^2)(b-c)}}{4a-(b-c)}.$$

10. Let length of floor $= x$.

„ breadth „ $= y$.

$$\therefore \overline{x+2}\ \overline{y-1}, \text{ or } xy+2y-x-2 = xy, \text{ or } -x+2y = 2,$$

$$\overline{x-4}\ \overline{y+3}, \text{ or } xy-4y+3x-12 = xy, \text{ or } 3x-4y = 12;$$

whence $x = 16$, $y = 9$.

11. (1) and (2) Arts. 210, 211, 223.

(3) See art. 208, $\frac{BC^2}{AB^2} = \frac{BC^2}{AC^2+BC^2} = \frac{(BC \div AC)^2}{1+(BC \div AC)^2}$;

or $\sin^2\theta = \frac{\tan^2\theta}{1+\tan^2\theta}$ and $\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$.

(4) Note A, Case 4, p. 242.

12. (1) Arts. 234, 235.

(2) $\frac{ab \sin\theta}{2}$. For solution, see Art. 264.

13. (1) Art. 283, (III.).

(2) For 1st equation when $x = 0$, $y = 2$, and when $y = 0$, $x = -3$. Follow instructions given in Art. 290.

In 2nd equation, $x = \pm y$, giving two lines intersecting in the origin, and inclined to the axis of x at angles of 45° and 135° respectively.

14. (1) Art. 314. (2) From the question, $y^2 = 4ax$; $c = y - bx$, we have by multiplying the left members and the right members of these equations together,

$$cy^2 = 4axy - 4abx^2 = 0,$$

$$\text{or } cy^2 - 4axy + 4abx^2 = 0.$$

This equation gives the coincident points of the straight line and the curve.

15. Art. 333.

16. Arts. 347 and 348.

1866. *July* 17*th*.

1. (1) $\frac{1}{20}$. (2) ·007.

(3) $(5+\sqrt{6})(\sqrt{3}-\sqrt{2}) = 3\sqrt{3}-2\sqrt{2} = 5{\cdot}196-2{\cdot}828$ $= 2{\cdot}368$.

2. (1) $\frac{479}{480}$. (2) £131 2*s*. 11$\frac{25}{81}$*d*.

3. (1) $\overline{a+b}-\overline{a-b}$ or $2b$, by actual multiplication.

(2) By multiplication we obtain

$a^2(p^2+q^2+r^2)+b^2(p^2+q^2+r^2)+c^2(p^2+q^2+r^2)=$

(upon the given assumption) 1. *Ans.*

(3) G.C.M. x^2+2x-1; $\frac{2x+1}{x-2}$ = the given fraction reduced to its lowest terms.

4. $\frac{2x+3y}{4a-5b} = \frac{3y+4z}{3b-a} = \frac{4z+5x}{2b-3a}$.

$\therefore$ (1) $(6b-2a)x+(24b-15a)y+(20b-16a)z = 0$.

(2) $(15b-5a)x+(9a-6b)y+(4b+8a)z = 0$.

(3) $(26a-29b)x+(9a-6b)y+(16a-20b)z = 0$.

(1) + (3) gives

$(24a-23b)x+(18b-6a)y = 0$. . . . . (4)

From (2) $\times(20b-16a)-(1)\times(4b+8a)$,

$$(300b^2-340ab+80a^2)x+(-120b^2+276ab-144a^2)y+0z=0.$$

$$\underline{(24b^2+40ab-16a^2)x+(96b^2+132ab-120a^2)y+0z = 0}$$

$$(276b^2+380ab+96a^2)x-(216b^2-144ab+24a^2)y = 0. \quad (5)$$

Eliminating x from (4) and (5),

$$(ab^2+20a^2b)y = 0$$

Eliminating y $\quad (ab^2+20a^2b)x = 0 \quad \therefore \begin{cases} y = 0 \\ x = 0 \\ z = 0 \end{cases}$

Substitution gives

$$\therefore 7x+5y+8z = 0.$$

5. (1) Art. 114 (II). $S = n\frac{a+l}{2}$. (2) See Ex. 7, p. 59. *Ans.* 15 or 10.

6. (1) Art. 165 (I). $P_r = m\,.\,\overline{m-1}\,.\,\overline{m-2}\ldots\overline{m-r+2}\,.\,\overline{m-r+1}$.
(2) $12.11.10.9.8.7.6.5.4.3.2.1 = 479001600$.
(3) $\frac{12.11.10.9.8.7.6.5.4.3.2.1}{12} = 39916800$.

7. Art. 170. Pres. v. of an annuity of £100 for 10 years,

$$= \frac{12{\cdot}5779}{{\cdot}628895} \times 100 = 772{\cdot}1737:$$

The p. v. of £1 due at the end of 2 years is ·907, and $772{\cdot}1737 \times {\cdot}907 = 700{\cdot}362 =$ £700 7*s.* 2¾*d.*

8. (1) Art. 193. The whole number positive, or negative, which precedes the decimal point of a logarithm. It is not generally printed because in Briggs' system it is so easily found.

(2) $\bar{4}$. The negative sign is usually written *over* the characteristic, instead of before it.

(3) Art. 195, 4°. (4) $2^3 \times 3^2 = 72 \;\therefore\; 3 \log 2 + 2 \log 3 = \log 72$, or $({\cdot}3010300 \times 3) + ({\cdot}4771213 \times 2) = 1{\cdot}8573326 = \log 72$ and $0{\cdot}8573326 = \log 7{\cdot}2$.

9. (1) Transforming $a\,x^2 + b\,x + c$ into

$$\frac{(2\,a\,x+b)^2 + 4\,a\,c - b^2}{4\,a},$$

there are 3 distinct cases, according as

$$b^2 \text{ is } >, =, \text{ or } < 4\,a\,c.$$

When $b^2 > 4\,a\,c$ the expression $a\,x^2 + b\,x + c$ has two real and differing roots, contained in the formula

$$\frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a},$$

and has always the same sign as a, except when x lies between those roots. See Course, p. 117, Let the reader &c.

(2) $x = 9$.

(3) $x + \sqrt{x^2 - a^2} = \frac{a^2}{2\,(x+a)}$

$$2\,x^2 + 2\,a\,x - a^2 = -2\sqrt{\{(x+a)^2(x^2-a^2)\}}$$

$$4\,x^4 + 8\,a\,x^3 - 4\,a^3\,x + a^4 = 4\,(x^4 + 2\,a\,x^3 - 2\,a^3\,x - a^4)$$

$$4\,a^3\,x = -5\,a^4$$

$$x = -\frac{5\,a}{4}.$$

(4) Dividing 1st equation by 2nd.

$$x^2-xy+y^2=7 \dots\dots\dots\dots (\text{I})$$

Squaring 2nd equation

$$x^2+2xy+y^2=16. \dots\dots\dots\dots (\text{II})$$

By subtraction $3xy=9$

$$\therefore xy=3 \text{ and } 2xy=6.$$

Hence from (II) $x^2+y^2=10$, and consequently

$$x^2-2xy+y^2=4. \dots\dots\dots\dots (\text{III})$$

Extracting root of both sides

$$\begin{aligned} & x-y=2 \\ \text{but } \quad & \underline{x+y=4} \\ \text{By addition, } \quad & 2x=6' \end{aligned}$$

$$\therefore x=3 \text{ and } y=1.$$

10. Let $x=$ price paid, then by Question

$$x:x-5::3\tfrac{1}{4}:3, \text{ or } 12x=13x-65.$$

$$\therefore x=65.$$

11. (1) To adapt fig. XVIII., p. 199, draw a right line from A to P (or suppose it drawn), and let $\angle PAM=A$, then $\angle PCM=2A$, by Eu. III. 20,

$$\sin A=\frac{PM}{AP}, \sin 2A=\frac{PM}{PC}.$$

Hence $\dfrac{\sin 2A}{2\sin A}=\dfrac{PM}{PC}\div\dfrac{2PM}{AP}=\dfrac{AP}{2PC}=\dfrac{AP}{CP+AC}$ but this is less than unity; because side AP is $<CP+AC$ (Eu. I. 20),

$$\therefore \sin 2A<2\sin A.$$

(3) $\tan A+\sec A=a$. First, $\dfrac{\sin A}{\cos A}+\dfrac{1}{\cos A}=a$;

$$\therefore \sin A+1=a\sqrt{1=\sin^2 A}$$

$$(\sin A+1)^2=a^2(1+\sin A)(1-\sin A)$$

$$1+\sin A=a^2(1-\sin A) \therefore \sin A=\frac{a^2-1}{a^2+1}.$$

12. (1) Art. 234. (2) Draw a triangle (ABC) having the angle $BCA=120°$, side $a=4$, $b=1$; then (Art. 243),

$$\tan\frac{A-B}{2}=\frac{4-1}{4+1}.\cot 60°=\frac{3}{5}.\frac{1}{\sqrt{3}}=\frac{3}{5\sqrt{3}}$$

$$\tan\frac{A+B}{2}\tan 30^\circ = \frac{1}{\sqrt{3}}, \text{ and } \tan\left(\frac{A+B}{2}+\frac{A-B}{2}\right) =$$

$$\tan A = (233)\,\frac{\dfrac{1}{\sqrt{3}}+\dfrac{3}{5\sqrt{3}}}{1-\dfrac{1}{5}} = \frac{\dfrac{8}{5\sqrt{3}}}{\dfrac{4}{5}} = \frac{2}{\sqrt{3}}.$$

$$\tan B = \frac{\dfrac{1}{\sqrt{3}}-\dfrac{3}{5\sqrt{3}}}{1+\dfrac{1}{5}} = \frac{\dfrac{2}{5\sqrt{3}}}{\dfrac{6}{5}} = \frac{1}{2\sqrt{3}}.$$

$$\because (215)\ \cot A = \frac{\sqrt{3}}{2}, \cot B = 3\sqrt{3}.$$

13. (1) Art. 239. (2) Art. 263, p. 172.

14. See Note A, Case 2, p. 241. (2) Let a, b, c, represent the three sides, any two of which are greater than the third.

By Art. 234, $\sin A : \sin B : \sin C :: \sin\frac{A}{2} : \sin\frac{B}{2} : \sin\frac{C}{2}$

$$:: \quad a \quad : \quad b \quad : \quad c.$$

$$\therefore \sin\frac{A}{2} + \sin\frac{B}{2} > \sin\frac{C}{2}.$$

15. (1) Let the equations to the two lines be $y = m x + b$, and $y = m' x + b'$. The angle ϕ between the two lines is the difference of the angles which they make with the axis of x. The tangents of these angles are m and m'; therefore (2. p. 197) the tangent of the required angle is

$$\tan\phi = \pm\frac{m-m'}{1+m m'}. \dots\dots\dots\dots (a)$$

The negative sign is obtained by taking the supplement of the angle.

(2) The required equations will be of the form $y-k = m(x-h)$; and if $m = \tan\beta$, we obtain $y-k = \frac{2m}{1-m^2}(x-h)$; and $y-k = \frac{m-m}{1+m^2}(x-h) = 0$.

The two lines required must be one on each side of the given line, and their tangents, $2\,m$, and $m-m$.

Their equations are,

$$y-k = \frac{\tan 2\,m}{1+\tan^2 m}(x-h)\,;\ y-k = \frac{\tan m - \tan m}{1+\tan^2 m}(x-h).$$

16. In fig. XVII. assume $AD = a$, $DC = b$, and the co-ordinates of P to be x and y, and the radius to be c, then

$$(x-a)^2+(y-b)^2=c^2;\ \text{and developing}$$
$$x^2+y^2-2\,a\,x-2\,b\,y+a^2+b^2-c^2=0.$$

In fig. XVIII. $a=c$, and $b=0$.

$\therefore\ x^2+y^2=2\,(a\,x+b\,y)$, the form required.

(2) First, $x^2+y^2-2\,a\,x-2\,b\,y=0$,

Solving, $(x-a)^2+(y-b)^2=a^2+b^2$.

Here the ordinates of the centre are a, b, and the radius is

$$\sqrt{a^2+b^2}.$$

Constructing the circle it will be seen to pass through the origin.

From the given equation, $y=c$, $x=-\dfrac{c}{m}$.

The equation to the required lines will be in the form $y=m\,x$, where m is the tangent the required line makes with the axis of x, or $m=\dfrac{y}{x}$, and is $-m$, for one line, and m, for the other. The equation to the required lines is $y=\pm\,m\,x$.

17. (1) Art. 321. (2) Art. 325 (III.)
(3) Ex. 2, p. 226. (4) Art. 336.

1867. *July* 16*th*.

1. (1) 8. (2) 6400.
2. $92\frac{1}{2}:740::3:£24$. *Ans.*
3. (1) 68·833. (2) $2x^2-3\,x\,y+4\,y^2$.
4. $\dfrac{x^8-x^4+1}{x^8+x^4+1}$.
5. (1) Art. 102. (2) By the question

$$\frac{p\,a+q\,b+r\,c+s\,d}{p\,a+q\,b-(r\,c+s\,d)}=\frac{p\,a-q\,b+r\,c-s\,d}{p\,a-q\,b-(r\,c-s\,d)}$$

$$\therefore\ \frac{p\,a+q\,b}{r\,c+s\,d}=\frac{p\,a-q\,b}{r\,c-s\,d}$$

$$\text{and}\ \frac{p\,a+q\,b}{p\,a-q\,b}=\frac{r\,c+s\,d}{r\,c-s\,d}.$$

$$\therefore\ \frac{p\,a}{q\,b}=\frac{r\,c}{s\,d}\ \text{and}\ \frac{a}{b}=\frac{c}{d}$$

$$\therefore\ \frac{p}{q}=\frac{r}{s}\ \text{and}\ p:q::r:s.$$

6. (1) Art. 166 (I).

(2) The permutations of the 20 consonants are $20 \times 19 = 380$; the number of words accruing therefrom $380 \times 5 = 1900$. (See Ex. 5, p. 95.)

7. (1) Art. 117 (I).

(2) If the pairs of terms taken are a and l, or

$$\left.\begin{array}{l} a, \text{ and } a\,r^{n-1} \\ a\,r \text{ and } a\,r^{n-2} \\ a\,r^2 \text{ and } a\,r^{n-3} \\ \text{\&c.} \quad \text{\&c.} \end{array}\right\}$$

then their products will be

$$\left.\begin{array}{l} a \times a\,r^{n-1} \\ a\,r \times a\,r^{n-2} \\ a\,r^2 \times a\,r^{n-3} \\ \text{\&c.} \quad \text{\&c.} \end{array}\right\} = a^2\,r^{n-1},$$

and the sum of their products

$$\frac{n}{2}\,a^2\,r^{n-1}.$$

8. (1) Let A be the required amount, p the annuity, r the amount, of £1 in 1 year, n the number of years. At the end of the 1st year p is due, at the end of 2nd year $r\,p+p$, is due or $(r+1)\,p$, at end of 3rd year $r\,(r+1)\;p+p$, or $(r^2+r+1)\,p$, and similarly at end of n years $(r^{n-1}+r^{n-2}+\ldots+1)\,p$

$$\therefore A = \frac{r^{n}-1}{r-1}\,p.$$

(2) Art. 176. The amount of any sum p, at compound interest for n years is, if $\frac{R}{100}$ (which is the interest of £1 for 1 year) $= r$, $p\,(1+r)^n$. By the question $2\,p = p\,(1+r)^n$ and

$$n = \frac{\log 2}{\log (1+r)}.$$

$$\text{Now } 1+r = 1{\cdot}04 = \frac{8 \times 13}{100}.$$

$$\therefore \log (1{\cdot}04) = \log 13 + 3 \log 2 - \log 100$$

$$\text{and } \frac{\log 2}{\log 1{\cdot}04} = \frac{{\cdot}30103}{{\cdot}0170334} = 17{\cdot}6735 \text{ years. } \textit{Ans.}$$

9. (1) $x = \frac{3}{2}$. (2) $x = 2$ or $3\frac{5}{21}$. (3) $x = 4$ or 3; $y = 3$ or 4.

10. Let $x =$ the cost price.

$$x : 24 :: 100 : 100+x.$$

$$\therefore x^2 + 100x = 2400; \text{ Completing the square, \&c.}$$

$$x = 20.$$

11. (1) Art. 211. (2) Art. 214.

(3) $$\sin^2\theta + \cos^2(90^\circ - \theta) = 1$$
$$2\sin^2\theta = 1$$
$$\sin\theta = \frac{1}{\sqrt{2}} \quad \therefore \theta = 45^\circ.$$

(4) $\tan\theta = 2\sin\theta$
$$\therefore \sin\theta = 2\sin\theta\sqrt{1-\sin^2\theta}$$
$$\therefore 1 - \sin^2\theta = \frac{1}{4} \text{ or } \sin^2\theta = \frac{3}{4}$$
$$\therefore \sin\theta = \frac{1}{2}\sqrt{3} \quad \therefore \theta = 60^\circ.$$

12. (1) Art. 235.

(2) This formula is not adapted to logarithmic computation when the angle is near 90°; but $\cos\frac{A}{2}$, $\sin\frac{A}{2}$ and $\tan\frac{A}{2}$ are so, see art. 238.

(3) $a = 10$, $b = 8$, $c = 6$; $2\,|\,24$, $12 = S$.

$S-a=2.$ $S-b=4.$ $S-c=6.$

$$\tan\frac{A}{2} = \sqrt{\frac{(S-b)\,.\,(S-c)}{S\,.\,(S-a)}}.$$

$$\therefore \log.\tan\frac{A}{2} = \frac{1}{2}\{(\log 4 + \log 6) - (\log 12 + \log 2)\} = 0.$$
$$\therefore \frac{A}{2} = 45^\circ \text{ and } A = 90^\circ.$$

13. (1) Art. 242. (2) Art. 265.

14. (1) Art. 233.

(2) Because the sum of the given angles equals two right angles, the three angles are the angles of a triangle. But by Art. 234 the sides of a triangle are proportional to the sines of their opposite angles; but a, b, c, being the three sides of a triangle (the three opposite angles of which $A+B+C = 180^\circ$), we have, by 265,
$$a^2 = b^2 + c^2 - 2\,b\,c\,.\cos A.$$
$$\therefore \sin^2 A = \sin^2 B + \sin^2 C - 2\sin B\,.\sin C\,.\cos A\,.$$

15. (1) Art. 289 (V.) (2) Ex. 4, p. 195.

16. (1) Art. 299. (2) Art. 303.

17. (1) Art. 308. (2) Ex. 2, p. 215.

1868. *July 21st.*

1. ·1. (2) 701. (3) $\cdot 339\dot{2}85714\dot{}$.

2. 3 : 200 :: 94½ : £6300 = the sum invested.

3. $$\frac{x^2-3x+2}{x+1}.$$

(2) $x^3+3\,a\,x\,y+y^3-a^3=\left\{x^2-xy+y^2+a(x+y+a)\right\}$
$(x+y-a)$.

(3) $x^3+3x+2=0$.

4. (1) Art. 144.

Let x be the quantity.

$$\frac{a+x}{b+x}=\frac{e}{f};\ af+fx=eb+ex;\ x=\frac{eb-af}{f-e},\ \text{Ans.}$$

$$\therefore\ \frac{a+\dfrac{eb-af}{f-e}}{b+\dfrac{eb-af}{f-e}}=\frac{e\,(b-a)}{f\,(b-a)}=\frac{e}{f}.$$

5. (1) See art. 118.

$$l=a\,r^{n-1}\ \therefore\ r^{n-1}=\frac{l}{a},$$

and since n is given, r^{n-1} and r^n and r can be found; then (117, I.)

$$S=a\,\frac{r^n-1}{r-1}.$$

(2) Let 1st term$=f$. $S_{40}=\dfrac{2f+39\,d}{2}\,.\,40$

$$=40f+780\,d=a;\ S_{50}=50f+1225\,d=b.$$

$$\therefore\ 50f+1225\,d-(40f+780\,d)=10f+445\,d=b-a.$$

$$\because\ S_{40}=a\ \therefore\ d=\frac{a-40f}{780};\ \text{and}\ \because\ S_{50}=b\ \therefore\ d=\frac{b-50f}{1225}$$

$$\text{whence } f=\frac{49\,a}{400}-\frac{39\,b}{500}.$$

$$\therefore\ \frac{49\,a}{40}-\frac{39\,b}{50}+445\,d=b-a.$$

$$\therefore\ d=\frac{b}{250}-\frac{a}{200},\ \text{or}\ \frac{1}{50}\left(\frac{b}{5}-\frac{a}{4}\right).$$

6. (1) Art. 166 (I).

$$C_r=\frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}\ldots\overline{n-r+2}\,.\,\overline{n-r+1}}{r\,.\,r-1\,.\,r-2\ldots\ldots\ldots\ldots\,3\,.\,2\,.\,1}.$$

(2) $26-5=21=$ the number of consonants in alphabet. (Ex. 5, p. 95) $21\times 20=420$ for the consonants, and with one vowel in each arrangement, and the vowel may occupy 3 positions, beginning, middle, and end, $420\times 5\times 3=6300$.

7. Art. 160.

8. (1) Log $\cdot 0027=\bar{3}\cdot 4313638$.

(2) First, $7^2\times 3^2\times 2^3=3528$. $\therefore$ log $3528=2\log 7+2\log 3+3\log 2=1\cdot 6901960+\cdot 9542426+\cdot 9030900=3\cdot 5475286$.

(2) See answer to Question 4 (3), p. 39 of this Key.

9. (1) $x = 1$. (2) $x = 1$.

(3) By adding the two equations, we have,

$$x^2+2\,x\,y+y^2 \text{ i.e. } (x+y)^2 = a+b \quad \therefore x+y = \sqrt{a+b}:$$

$$\text{But } (x+y)x = a \quad \therefore x = \frac{a}{x+y} = \pm\frac{a}{\sqrt{a+b}}$$

$$\text{and } (x+y)y = b \quad \therefore y = \frac{b}{x+y} = \pm\frac{b}{\sqrt{a+b}}.$$

10. Let x and y = the respective numbers,

$$\begin{array}{l} x+y = 99 \\ \underline{x-y = 45} \\ 2\,y = 54 \quad y = 27 \;\therefore x = 72. \end{array}$$

Or it might have been solved by taking $10x+y$, $10y+x$ for the numbers, then

$$\begin{array}{ll} 11x+11y=99 & \text{or } x+y=9 \\ 9x-\;\;9y=45 & \;,, \;\; x-y=5 \end{array}$$

whence $x=7$, $y=2$, and the numbers are 72 and 27 as before.

11. (1) The ratios for 60° are deduced from those of 30° as shown by arts. 227, 229.

(2) $\cos 60° = \frac{1}{2}$ $\cos 30° = \frac{\sqrt{3}}{2}$.

And if $A = 45°$, $B = 30°$, and $C = 15°$, then

$$\cos(A+B-C) = \frac{1}{2}, \text{ or } \cos(45°+30°-15°) = \frac{1}{2}$$

$$\cos(A-B+C) = \frac{\sqrt{3}}{2}, \text{ or } \cos(45°-30°+15°) = \frac{\sqrt{3}}{2}$$

$$\cos(A+B) = \sin C \text{ or } \cos(45°+30°) = \sin 15°.$$

12. (1) Art. 258. (2) Art. 258, fig. X.

Suppose $\alpha = 30°$.
$\beta = 60°$.
$a = 100$ feet.

and let x = breadth of river.
y = height of tree.

$$y = a \,.\, \sin\alpha \,.\, \sin\beta \,.\, \text{cosec} \,.\, (\beta-\alpha).$$

$$\begin{aligned}
\log 100 &= 2{\cdot}000000 \\
\log \sin 30^\circ &= \bar{1}{\cdot}698970 \\
\log \sin 60^\circ &= \bar{1}{\cdot}937531 \\
\log \operatorname{cosec} 30^\circ &= \ {\cdot}301030 \\
&\overline{1{\cdot}937531} = \log \text{ of } 86{\cdot}6 = y.
\end{aligned}$$

$x = a \sin \alpha \,.\, \cos \beta \,.\, \operatorname{cosec} (\beta - \alpha)$.

$$\begin{aligned}
\log 100 &= 2{\cdot}000000 \\
\log \sin 30^\circ &= \bar{1}{\cdot}698970 \\
\log \cos 60^\circ &= \bar{1}{\cdot}698970 \\
\log \operatorname{cosec} 30^\circ &= \ {\cdot}301030 \\
&\overline{1{\cdot}698970} = \log \text{ of } 50 = x.
\end{aligned}$$

$\therefore$ height of tree $= 86{\cdot}6$ feet, width of river 50 feet.

13. (1) By art. 234, the sides are proportional to their opposite angles,

$$\therefore \frac{b-c}{b+c} = \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{\tan \frac{B-C}{2}}{\tan \frac{B+C}{2}}. \quad \text{See Art. 243.}$$

$$i.e.\ (b-c) : (b+c) :: \tan \frac{B-C}{2} : \tan \frac{B+C}{2}.$$

(2) $\log \tan \frac{B-C}{2} = \log (b-c) + \log \cot \frac{A}{2} - \log . (b+c)$.

Given $b = 17$, $c = 7$, $A = 60^\circ$.

$B + C = (180^\circ - 60^\circ) = 120^\circ$.

$b - c = 10$, $b + c = 24$

$\frac{A}{2} = 30^\circ$, $\frac{B+C}{2} = 60^\circ$.

$$\begin{aligned}
\log b - c &= 1{\cdot}0000000 \\
\log \cot \frac{A}{2} &= {\cdot}2385606 \\
&\overline{1{\cdot}2385606} \\
\log (b+c) &= \underline{1{\cdot}3802113} \\
&\bar{1}{\cdot}8583593
\end{aligned}$$

$= \log \tan \frac{B-C}{2} \quad \therefore \frac{B-C}{2} = 35^\circ\ 49'$.

$$\begin{aligned}
\tfrac{1}{2}(B+C)\ & 60^\circ \\
\tfrac{1}{2}(B-C)\ & \underline{35^\circ\ 49'} \\
& \overline{95^\circ\ 49'} = B. \\
& \overline{24^\circ\ 11'} = C.
\end{aligned}$$

To form the logs required for the solution above from those given in the paper, the candidate would have to remember that

$\cos 60° = \sin 30° = \frac{1}{2}$; and $\cos 30° = \sin 60° = \frac{\sqrt{3}}{2}$.

$$\text{Then } \cot\frac{A}{2} = \cot 30° = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \quad \log \ \cot\frac{A}{2} = \cdot 2385606.$$

For log of $(b+c)$ or log of 24, $\because 2^3 \times 3 = 24$,

$$\therefore \log 24 = 3 \log 2 + \log 3 = 1\cdot 3802113.$$

In a similar way the logs required to solve Question 12 may be found.

14. (1) Art. 237.

(2) Let the sides be $x-3$, x, and $x+3$; then (Art. 237).

$$\frac{(x-3)+x+(x+3)}{2} = \frac{3\,x}{2} = S,$$

and for the equilateral triangle

$$\frac{x+x+x}{2} = \frac{3\,x}{2} = S; \ \therefore \text{ by the question,}$$

$$\sqrt{\frac{3\,x}{2} \cdot \frac{x+6}{2} \cdot \frac{x}{2} \cdot \frac{x-6}{2}} = \frac{4}{5}\sqrt{\frac{3\,x}{2} \cdot \frac{x}{2} \cdot \frac{x}{2} \cdot \frac{x}{2}}.$$

$$\sqrt{\frac{3\,x^4 - 108\,x^2}{16}} = \frac{4}{5}\sqrt{\frac{3\,x^4}{16}}$$

$$\frac{3\,x^4 - 108x^2}{16} = \frac{16}{25} \cdot \frac{3\,x^4}{16} = \frac{3\,x^4}{25}.$$

$$27\,x^4 - 2700x^2 = 0; \ x^2 = 100, \text{ and } x = 10.$$

$$\therefore \left.\begin{aligned} x-3 &= 7 \\ x &= 10 \\ x+3 &= 13 \end{aligned}\right\}$$

15. Art. 283. (2) Art. 294.
16. Art. 300 (V). (2) Art. 306.
17. Art. 327 (IV). (2) Ex. 3, p. 227.

1869. *July 20th.*

1. (1) Art. 81. (2) ·9525. (3) 33 seconds.
2. (1) $\frac{1}{2} : \frac{1}{3} : \frac{1}{4} :: £12\ 1s.\ 6d. : £8\ 1s. : £6\ 0s.\ 9d.$

   (2) 4·1231.

3. (1) 1. (2) 2.

(3) By substitution, and actual division, $x^2+2\,b\,x-3\,b^2 \div x-b = x+3b$; and $x^2-2\,b\,x-3\,b^2 \div x+b = x-3\,b$.

4. (1) If $a:b::b:c:c:d$, prove $a:d::a^3:b^3$.

First $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$ multiplying by the equal quantities $\frac{a}{b}$ and $\frac{c}{d}$, then $\frac{a^2}{b^2}=\frac{b}{d}$, and again, multiplying by $\frac{a}{b}$, we have $\frac{a^3}{b^3}=\frac{a}{d}$,

$$\therefore a:d::a^3:b^3.$$

(2) One quantity varies directly as a second, and inversely as a third, where it varies jointly as the second and the reciprocal of the third.

If $A=\frac{m\,B}{C}$, A varies directly as B, and inversely as C.

(3) In the one case $B=1\frac{1}{2}$, and in the other $B=3$.

5. (1) Art. 114 (II.)

(2) P. 62, Ex. 9. Let a, $a\,r$, $a\,r^2$, $a\,r^3$, be the four terms of the series. $\frac{a\,r^3}{a}=r^3 \therefore \frac{1}{3\sqrt{3}}=r^3$, and $r=\frac{1}{\sqrt{3}}$;

$$\text{and } S=3\,\frac{1}{\sqrt{3}-1}=\frac{3}{2}(\sqrt{3}+1).$$

6. (1) P. 96 (III). (2) $\frac{10.9.8.7.6}{1.2.3.4.5}\cdot\frac{10.9.8.7.6}{1.2.3.4.5}=63504.$

7. Art. 169. Present value $= £A\,\dfrac{1-\left(\frac{1}{1+r}\right)^n}{r}$.

8. (1) $x=\frac{a+b}{a\,b}$; $y=\frac{b^3-a^3}{a\,b}$.

(2) $x=\frac{1-a}{2\,a}$ or $-\frac{1+a}{2\,a}$.

(3) $x=2$; $y=\frac{1}{2}$.

9. Let x and $x+10$ = breath and length, then $x.(x+10)=1131$, or $x^2+10x=1131$, a quadratic, and solving, $x=29$, $\therefore$ 39 yds. = length; 29 yds. = breadth.

10. Art. 189. (2) Page 128, 4°.

(3) First, $\log 5 = \log 10 - \log 2 = 1 - \cdot 30103 = \cdot 69897$,

and $\log\ 2^{-\frac{1}{5}} = \log \frac{1}{\sqrt[5]{2}} = \log 1 - \frac{1}{5} \log 2 = \cdot 000000 - \cdot 060206$

$= \bar{1} \cdot 939794$.

(4) $\log \cdot 00002 = \bar{5} \cdot 30103$.

(5) First, $62 \cdot 5 = \frac{5^4}{10} = 4 \log 5 - 1 = 2 \cdot 79588 - 1 = 1 \cdot 79588$.

(6) $\log\ 5^{-\frac{1}{2}} = \log \frac{1}{\sqrt{5}} = \log 1 - \frac{1}{2} \log 5$

$$= \cdot 000000 - \cdot 349485 = \bar{1} \cdot 650515.$$

11. Let the known angle be a (A, in 220 &c.), then the general formula is $\sin a = \sin (\pi - a) = \sin (2 n \pi + a)$.

(2) $\sin x + \sqrt{1 - \sin^2 x} = 1$

$$1 - \sin^2 x = 1 - 2 \sin x + \sin^2 x,\ 2 \sin^2 x - 2 \sin x = 0$$

$$\sin x = 1$$

$$\therefore\ x = 90;$$

or, $\cos x + \sqrt{1 - \cos^2 x} = 1$; which gives $x = 0$.

(3) $\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x$,

$$\sin^2 x = 0$$

$$\therefore\ x = 0;$$

$$\text{or, } 2 \cos^2 x - 1 = \cos^2 x$$

$$\cos^2 x = 1$$

$$\therefore\ x = 0.$$

12. (1) Let $\frac{A+B}{2} = \phi$ and $\frac{A-B}{2} = \theta$.

$$\text{then } A = \frac{A+B}{2} + \frac{A-B}{2} = \phi + \theta,$$

$$\text{and } B = \frac{A+B}{2} - \frac{A-B}{2} = \phi - \theta.$$

$\cos A = \cos (\phi + \theta) = \cos \phi \cos \theta - \sin \phi \sin \theta$ (232).

$\cos B = \cos (\phi - \theta) = \cos \phi \cos \theta + \sin \phi \sin \theta$.

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}.$$

The last equation is usually written

$$\cos B - \cos A = 2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}.$$

(2) Ex. 19, p. 146.

(3) $\cos^2 A + \sin^2 A = 1$

$2 \sin A \, . \cos A = \sin 2 A$, Ex. 3, p. 141.

$\therefore \cos^2 A + 2 \sin A \, . \cos A + \sin^2 A = 1 + \sin 2 A$;

and $\cos^2 A - 2 \sin A \, . \cos A + \sin^2 A = 1 - \sin 2 A$.

$\therefore \cos A + \sin A = \pm \sqrt{1 + \sin 2 A}$.

and $\cos A - \sin A = \pm \sqrt{1 - \sin 2 A}$.

13. Art. 266.

14. (1) Art. 243.

(2) See fig. IV. p. 147. $B C = B D + D C = A B \cos B + A C \cos C$, or $a = c \cos B + b \cos C$.

The same result is obtained from fig. V., and similarly from both,

$$b = a \cos C + c \cos A,$$
$$c = b \cos A + a \cos B.$$

Multiplying each member of these three equations by a, b, and c, respectively, and adding and dividing by 2, we obtain the required equation,

$$\tfrac{1}{2} (a^2 + b^2 + c^2) = b \, c \cos A + c \, a \cos B + b \, a \cos C.$$

15. Ex. 5, p. 196.

16. See Ex. 1, p. 204.

17. The sum of the focal distances of any point P, on an ellipse, is constant (art. 333).

1870. *July 19th.*

1. Art. 41. (2) $\frac{157}{225}$.

2. (α) 695·701, &c. yds. (β) ·333, &c.

The remainders are composed of a regularly increasing series of the same digit 2.

3. (α) $\dfrac{8 \, a \, b \, c}{a \, (b^2 + a \, c - a \, b - c^2) + b \, (c^2 - b \, c)}$.

(β) -4.

4. (α) See Questions (4) and (8), pages 84 and 85.

When one quantity varies as both of two others jointly, it means that if either the second or third remain constant, the first varies as the other. The value of a quantity of goods varies jointly as the *number* of the articles, and the *price* of each. At a given *price* per article the whole value varies as the *number* of them. On the other hand, for a given *number* of articles the whole value varies as the *price* of one.

(β) Let x = number outside, y = number inside, a year ago; then $x \,.\, 1{\cdot}025 + y \,.\, 1{\cdot}025 = x \,.\, 1{\cdot}04 + y \,.\, {\cdot}89$.

$$x \,.\, {\cdot}015 = y \,.\, {\cdot}135$$

$$\frac{x}{y} = \left(\frac{{\cdot}135}{{\cdot}015}\right) = \frac{9}{1}.$$

5. (a) The successive triangles $(1+3), (1+3+5), (1+3+5+7)$, &c. are all composed of n terms of the odd numbers (Ex. 4, p. 59), and the formula gives $S = n^2$, therefore the number of men in every triangle was a square number, and the solid triangle could always be transformed into a solid square.

(β) Let the three squares be b^2, $25\,b^2$, $49\,b^2$, which are evidently in arithmetical progression.

The roots are b, $5\,b$, $7\,b$.

Sums taken in pairs, $6\,b$, $8\,b$, $12\,b$.

$$\text{Reciprocals } \frac{1}{6\,b}, \frac{1}{8\,b}, \frac{1}{12\,b},$$

$$\text{or } \frac{4}{24\,b}, \frac{3}{24\,b}, \frac{2}{24\,b},$$

which are in arithmetical progression.

Let b have any numerical value, say $b = 1$,

then 1, 25, 49, are in A.P.

and $\frac{1}{6}, \frac{1}{8}, \frac{1}{12}$ " " "

6. Let 3 be Protestants, and 4 Catholics.

$$\frac{20 \,.\, 19 \,.\, 18}{3 \,.\, 2 \,.\, 1} \times \frac{15 \,.\, 14 \,.\, 13 \,.\, 12}{4 \,.\, 3 \,.\, 2 \,.\, 1} = 1556100.$$

7. Art. 169. Present value $= A \dfrac{1 - \left(\frac{1}{1+r}\right)^n}{r}$. In this instance n is infinite $\therefore$ p. v. $= \dfrac{225}{{\cdot}035} =$ £6428 11s. 5$\frac{1}{7}d$.

8. (a) $x = \dfrac{a\,b\,c}{a\,b + b\,c - c\,a}$, $y = -\dfrac{a\,b\,c}{a\,b + b\,c - c\,a}$.

(β) $x = \dfrac{1 \pm \sqrt{5}}{2}$.

(γ) $y = \dfrac{12}{x}$ $\therefore x^5 - \dfrac{12^5}{x^5} = 781$;

$$\text{or } x^{10} - 248832 = 781\,x^5.$$

$$\therefore x^5 = \frac{781 + \sqrt{609961 + 995328}}{2}$$

$$= \frac{781 \pm 1267}{2} = 1024 \text{ or} -243.$$

$$\left.\begin{aligned} \therefore x &= 4 \text{ or} -3 \\ y &= \frac{12}{x} = 3 \text{ or} -4 \end{aligned}\right\}.$$

9. (α) Let x = whole distance, y no. of miles per hour, or full speed, then $\frac{3y}{4}$ = diminished speed, and $x-y$, $x-60$, = remainders of journey upon the two suppositions.

$$1 + \frac{4x-4y}{3y} = \frac{x}{y} + 1\tfrac{1}{3}, \text{ or } 6y + 8x - 8y = 6x + 8y,$$

$$\frac{60}{y} + \frac{4(x-60)}{3y} = \frac{x}{y} + 1\frac{1}{6}, \text{ or } 360 + 8x - 480 = 6x + 7y.$$

$$\begin{aligned} 2x - 10y &= 0 \\ 2x - 7y &= 120 \\ \hline 3y &= 120 \\ y = 40 \text{ and } x \text{ (by substitution)} &= 200. \end{aligned}$$

(β) [1] The information would have been correct. [2] Yes.

10. (1) To the base 3, the logs of the numbers 729 and 2187 are 6 and 7; therefore the log of 2000 must be between 6 and 7, and the characteristic will be 6.

(2) $\cdot 9754778 - \cdot 9754318 = \cdot 000046$.

$1000 : 666 :: \cdot 000046 : \cdot 000030636$;

and $\cdot 9754318 + \cdot 0000306 = \cdot 9754624$, and the complete logarithm required $= 6 \cdot 9754624$.

11. (1) Ex. 5, p. 184.

(2) Assume in the ordinary way, A for the origin, and the co-ordinates $x_1 = 0, y_1 = 7$; and $x_2 = -3, y_2 = 5$; then B will have the co-ordinates $x = 4, y = 4$.

The position of B may be any imaginable point in any of the four right angles (Art. 216) according to the value and sign assumed for the co-ordinates of the adjacent vertices.

12. Ex. p. 201.

(3) See fig. XXXIII., $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is (327 IV.) the equation to an ellipse with (the centre) C, for the origin of co-ordinates. In this case a is the line $C\,A'$ and b the line $C\,B$,

$$\text{and } a\,.\cos\alpha = \cos^2\alpha.$$

$$b\,.\sin\alpha = \sin^2\alpha.$$

$$\therefore\ a\,.\cos\alpha + b\,.\sin\alpha = \frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

O being the origin, $a\,.\cos\alpha$ gives the point A', and $b\,.\sin\alpha$, the point B.

13. Arts. 208, 217.

14. (1) Art. 234. $\dfrac{\sin A}{\sin B}=\dfrac{a}{b}$

$$\therefore\ \sin A = \sin B\frac{a}{b}, \text{ and } \frac{\sin A}{a}=\frac{\sin B}{b}.$$

Again (234) $\dfrac{\sin B}{\sin C}=\dfrac{b}{c}$

$\therefore\ \sin B = \sin C\dfrac{b}{c}$, and $\dfrac{\sin B}{b}=\dfrac{\sin C}{c}$;

$\therefore \dfrac{\sin A}{a}=\dfrac{\sin B}{b}=\dfrac{\sin C}{c}.$

(2) Art. 243.

15. Art. 268.

1871. *July* 18*th*.

1. (1) 10296.

(2) It is shown in art. 41—the letters representing any and every expression—that every divisor of a and b, divides $a-p\,b$ that is c. Similarly, every divisor of b and c also divides c and d. Therefore, every divisor of a and b also divides d.

2. (1) The same time that one train would take to pass a fixed point, that is, the time either required to travel 44 yards.

$$52800 : 44 :: 3600 : 3 \text{ seconds.}$$

(2) In other words, reduce 12 inches to the fraction of a French mètre; $\dfrac{12}{39\cdot37}=\cdot305.$

3. $(1\cdot05)^5\times 32{,}000{,}000 = 40{,}340832.$

4. (1) $a\,b\,x\,y$. (2) $1+2x+x^2+6x^3+9x^4$.

5. (1) Art. 141. $\dfrac{a}{b}=\dfrac{c}{d}$, divide unity by each of these equals, then $\dfrac{b}{a}=\dfrac{d}{c}$, *i.e.*, $b : a :: d : c$.

(2) Art. 148.

6. (1) $P_n = n(n-1)(n-2) \ldots 3 \cdot 2 \cdot 1$; or take the product of the numbers 1, 2, 3, &c., up to and including n.

(2) and (3) In each case $n(n-1)$ times.

7. (1) Ex. 12, p. 63. $r = -\frac{1}{2}$, $a = 1$ $\therefore S = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$.

(2) The odd terms all end with a *plus*, making the sum *greater*, and all the even terms end with a *minus*, making the sum *less*, than the sum of the terms to infinity; and this alternate excess and deficiency is the law of the series.

8. (α) $x = 2$ or $\frac{1}{2}$. (β) [1] $x = \frac{-a(a+2)}{a-2}$; $y = \frac{2a(a-1)}{a-2}$.

[2] Yes. $a = 1, -2, 0$, or ∞.

9. See Ex. 3, p. 101.

Present value of a perpetual annuity $= \frac{p}{r}$.

do of annuity for n years $= \frac{1-\left(\frac{1}{1+r}\right)^n}{r} p$.

$\therefore$ the value in reversion is the difference of these quantities,

$$\frac{p}{r} \cdot \frac{1}{(1+r)^n}.$$

(2) First, $\frac{1}{(1+r)^n} = (1+r)^{-n}$, see 82 ($\beta$), and

$$\therefore \frac{p}{r} \cdot (1+r)^{-n} : \frac{p}{r} :: (1+r)^{-n} : 1;$$

10. (1) Art. 189. (2) Art. 192. The advantage of Briggs' system consists in the circumstance that the mantissa does not depend upon the position of the decimal point, but upon the significant figures.

(3) Page 28, 4°.

(4) The logarithm of the required number $= \frac{1}{2} \log 10$,

or $\log N = \frac{\log 10}{2}$, $\therefore N = \sqrt{10} = 3 \cdot 1622$, &c.

11. (1) Let H (Art. 294, fig. XIII.) be the point (ξ, η), $D\,E$ the given line, and $y = m\,x + b$ its equation, &c., &c.

(2) The parabola. Art. 307.

12. Art. 297.

13. Ex. 2, p. 204.

14. Ex. 1, p. 226.

(2) *Four* such tangents can be drawn, and the area they enclose is equal to the rectangle contained by the major and minor axes of the ellipse. See art. 338.

15. (1) Art. 223.

(2) The limit of the Euclidean angle is 180°.

(3) Art. 233 (2).

16. Art. 243.

(2) Art. 242.

17. The angle of depression to the opposite side of the river being 45°, its breadth is equal to the height of the higher cliff, 200 ft.

To find the height of the lower cliff,

$$\tan 30° = \frac{1}{\sqrt{3}}$$

∴ the perpendicular line of depression from the horizon

$$= \frac{200}{\sqrt{3}} = 115 \text{ ft. } 6 \text{ in., and } 200 - 115\cdot\dot{5} = 84 \text{ ft. } 6 \text{ in.}$$

July 16*th*. 1872.

1. (1) Arts. 92 and 93.

(2) $365\cdot25 - 365\cdot242264 = \cdot007736$ error in one year; $\cdot7736 =$ error in one century; then $263\tfrac{3}{125} = 263\cdot024$;
$263\cdot024 \div \cdot7736 = 340$, the number of centuries required.

2. (1) Art. 42, G. C. M., 174. (2) G. C. M., 406.

(3) Art. 43, G. C. M., 58. (4). L. C. M., 10·10625.

3. $\sqrt{\frac{8820}{605}} = \sqrt{\frac{1764}{121}} = \frac{42}{11} = 3\frac{9}{11}$.

(2) $\frac{7\sqrt{10} - 5\sqrt{5} + 8\sqrt{2} - 22}{1}$.

4. (1). 1.

(2). $\frac{\overline{x-a}(x^3+a^3) - (x+a)(x^3-a^3)}{(x+a)\ (x^3+a^3)}$

$$= \frac{(x-a)\ (x+a)\ (2\,a\,x)}{(x+a)\ (x+a)\ (x^2-a\,x+x^2)} = \frac{2\,a\,x(x-a}{x^3+a^3};$$

$$\frac{x+a}{x-a} + \frac{x^2+a^2}{x^2-a^2} = \frac{2(x^2+ax\ a^2)}{x^2-a^2}.$$

$$\frac{\dfrac{2\,a\,x\,(x-a)}{x^3+a^3}}{\dfrac{2(x^2+a\,x+a^2)}{x^2-a^2}}$$

$$=\frac{a\,x\,(x+a)\,(x-a)^2}{(x^2+a\,x+a^2)\,(x^2-a\,x+a^2)\,(x+a)}=\frac{a\,x\,(x-a)^2}{x^4+a^2x^2+a^4}.$$

5. If $a : b :: c : d$, and $c : b :: b : d$, prove that $a : c :: c : b$.

$$\text{First,}\quad \frac{a}{b}=\frac{c}{d}\text{ and }\frac{c}{b}=\frac{b}{d}\text{ or (141) }\frac{b}{c}=\frac{d}{b};$$

$$\therefore \frac{a}{b}\times\frac{b}{c}=\frac{c}{d}\times\frac{d}{b}\text{ or }\frac{a}{c}=\frac{c}{b}\text{ or }a : c :: c : b.$$

6. (1) Arts. 165 (I.) and 166 (I.).

(2) $17\times16=272$ permutations of consonants taken 2 at a time. The number of words will be,

$$272\times5\times4=5440.$$

7. $S=a\,\dfrac{r^n-1}{r-1}$ (2) $\dfrac{3}{10}$, see Ex. 11, p. 63.

8. (1) An annuity for ever at 4 per cent. is worth 25 years' purchase, and (172) present value of an annuity at £1 for 3 years at 4 per cent. is, 2·775, and

$$\frac{5000}{2{\cdot}775}=1801{\cdot}\dot{8}0\dot{1}=\pounds1801\ 16\tfrac{4}{111}s.$$

(2) Art. 120, VII.

$$a=10{,}000,\ r=\frac{5}{7}$$

$$S=10{,}000\times\frac{7}{2}=35{,}000.\quad \pounds35{,}000\ \mathit{Ans.}$$

$$9.\quad \frac{1}{y}-\frac{1}{x}=\frac{1}{5}\ \ldots\ldots\ldots\ldots\ (1)$$

$$-\frac{1}{x}+\frac{1}{z}=\frac{2}{3}\ \ldots\ldots\ldots\ldots\ (2)$$

$$\frac{1}{y}\qquad+\frac{1}{z}=\frac{3}{2}\ \ldots\ldots\ldots\ldots\ (3)$$

$$(3)-(2)\quad \frac{1}{y}+\frac{1}{x}=\frac{5}{6}\ \ldots\ldots\ldots\ldots\ (4)$$

$$\left.\begin{array}{lll} (4)+(1) & \frac{2}{y}=1 & \therefore y=2 \\ (4)-(1) & \frac{2}{x}=\frac{2}{3} & \therefore x=3 \\ (3) & \frac{1}{2}+\frac{1}{z}=\frac{3}{2} & \therefore z=1 \end{array}\right\} \textit{Ans.}$$

$$(2) \qquad \overline{x-d}+x+x+d = 3x = 33,$$

$$\therefore x = 11, \text{ the middle term;}$$

$$\text{and } (11-d)^2+121+(11+d)^2 = 435;$$

$$\therefore 2d^2+363 = 435,$$

$$\text{and } d = \pm 6, \textit{ Ans.}$$

10. (1) 11; see Art. 189, and 1870, Qu. 10 and answer, p. 72 of this Key.

(2) $\log_{\sqrt[5]{10}} \sqrt[5]{10} = 1 \therefore \log_{\sqrt[5]{10}} 10 = 5$, and $\log_{\sqrt[5]{10}} 3$
$= \cdot 4771213 \times 5 = 2\cdot 385606.$

$$(3) \qquad \frac{864}{648}\times 3^4 = 4\times 3^3 = 108.$$

$$\log 108 = \cdot 6020600 + 1\cdot 4313639 = 2\cdot 0334239.$$

11. Ex. 4, p. 183.

12. The equation to the tangent is, $a^2 y y^1 + b x x^1 = a^2 b^2$.

Now at T (Fig. XXXIII), $y = O$.

$$\therefore x = \frac{a^2}{x'}, \text{ i.e. } OT = \frac{OA^2}{OM}.$$

The equation to the normal is,

$$y-y' = \frac{a^2 y'}{b^2 x'}(x-x').$$

At R, $y = O$. $\therefore x-x' = \frac{b^2 x'}{a^2}$, and $x = x'\left(1-\frac{b^2}{a^2}\right) = e^2 x'$,

$$\text{i.e. } CR = e^2 \,.\, CM.$$

(2) From the above $CT\times CR = a^2-b^2$, the difference of the squares of the semi-axes, as required.

13. (1) Case 2, p. 241.

(2) First, $\sin 15° = \sin (45°-30°)$ Arts. 227, 228

$= \sin 45° \cos 30° - \cos 45° \sin 30° = \frac{\sqrt{3}-1}{2\sqrt{2}}$; similarly,

$\cos 15° = \cos (45°-30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

But $\sin 75° = \cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$;

and $\cos 75° = \sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

14. (1) Art. 241.

(2) Let $a = 9$, $b = 7$, and $c = 4$,
then Art. 236;

$$S = 10,\ S-a = 1,\ S-b = 3,\ S-c = 6;$$

and Area $= \sqrt{10 . 1 . 3 . 6} = \sqrt{180}$. $\therefore \sin A = \frac{3}{7}\sqrt{5}$;

$$\sin B = \frac{\sqrt{5}}{3};\ \sin C = \frac{4}{21}\sqrt{5}.$$

1873. *July 23rd.*

1. (1) Arts. 92 and 93. (2) 1·2823. (3) $\cdot 01373\dot{8}$.

2. (1) $\frac{1429657}{19999998}$. (2) $9\cdot\dot{9}9000\dot{9}$.

(3) Art. 88.

(4) No decimal if squared will give a whole number, and if the decimal circulated, it would be equivalent to some definite quantity.

(5) Circulating decimals with exact roots only can have circulating square roots, as $6\frac{2}{3}$ or $6\cdot 66\dot{6}$ is the square root of $44\frac{4}{9}$ or $44\cdot\dot{4}$; and ·333 &c. is the square root of ·111 &c.

3. (1) 2·5173. (2) $x^2 + \frac{1}{2}x + \frac{1}{2} + \frac{1}{x}$.

4. (α) 0. (β) $\frac{x^3+x}{2}$ (γ) $12\,b^3$.

5. (1) $x = 1\frac{1}{3}$.

(2) Affected quadratic (art. 178), $x = 5$ or $5\frac{1}{2}$.

(3) Designating the three given equations by the letters (a) (b) and (c); we obtain,

From $2(b)-(a)$ $3y+z=2b-a \therefore z=2b-a-3y$.
" $(b)-(c)$ $y-z=b-c$ " $z=-b+c+y$

$$\therefore 2b-a-3y=y-b+c, \text{ and } y=\frac{3b-c-a}{4}$$

$$\text{and by substitution } z=\frac{3c-a-b}{4}$$

$$\text{" " " } x=\frac{3a-b-c}{4}.$$

6. (1) Let the three numbers of the arithmetical series be $(1+a^2-b)$, $(1+a^2)$, $(1+a^2+b)$. To find b in terms of a, $(1+a^2-b)(1+a^2+b)=(1-a^2)^2$ in the geometrical series.

$$4a^2=b^2$$
$$2a=b.$$

$\therefore$ the numbers are $1-2a+a^2$, $1+a^2$, $1+2a+a^2$;
and $1-2a+a^2$, $1-a^2$, $1+2a+a^2$.

(2) Let a be the first term, then the series will be

$$a+a^2+a^3, \text{ and } a=\frac{9}{10}(a^2+a^3)$$

$$\therefore 9a^2+9a=10, \text{ or } a^2+a=\frac{10}{9};$$

$$\text{and solving, } a=\frac{2}{3} \text{ or } -\frac{5}{3}.$$

The numbers are $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}$; or $-\frac{5}{3}, \frac{25}{9}, -\frac{125}{27}$.

7. (1) Art. 166, II. (2) 2520.

8. Art. 152. (2) If A varies as B^2, B^3 as C^4, &c.

then $\frac{A}{B^2}, \frac{B^3}{C^4}, \frac{C^5}{D^6}, \frac{D^7}{E^4}$ are constant, and their product is constant.

$$\therefore \frac{A}{B^2}\times\frac{B^3}{C^4}\times\frac{C^5}{D^6}\times\frac{D^7}{E^4}=\frac{A\,.\,B\,.\,C\,.\,D}{E^4},$$

$$\text{or } \frac{A}{E}\times\frac{B}{E}\times\frac{C}{E}\times\frac{D}{E},$$

which is constant, and does not vary at all.

9. For $\frac{R-1}{100}$ write r. Present value of an immediate annuity of £1 for n years (p. 100) is

$$\frac{1-\left(\frac{1}{1+r}\right)^n}{r}; \text{ deferred for } n \text{ years (171), } \frac{1-\left(\frac{1}{1+r}\right)^n}{r}\left(\frac{1}{1+r}\right)^n$$

$$=\frac{(1+r)^n-1}{r(1+r)^{2n}},$$

and the deferred annuity is to be equal in value to the immediate annuity of p, pounds

$$\therefore p\,\frac{1-\left(\frac{1}{1+r}\right)}{r} = \frac{(1+r)^n-1}{r\,(1+r)^n},$$

$$p = \frac{\dfrac{(1+r)^n-1}{r\,(1+r)^{2n}}}{\dfrac{(1+r)^n-1}{r(1+r)^n}} = \frac{1}{(1+r)^n};$$

$$\text{where } r = \frac{R-1}{100}.$$

10. (1) Qu. 10 (5) 1869, p. 69 of this Key, log $62{\cdot}5 = 1{\cdot}79588$, which by a necessary change of the characteristic, gives $\bar{\bar{3}}{\cdot}79588$, the logarithm required.

(2) $2^3 \times 3 = 24 \;\therefore\; 3\log 2 + \log 3 = 1{\cdot}3802113 = \log 24$:

$$\log\frac{1}{24} = \log 1 - \log 24 = \bar{2}.6197887.$$

(3) $\log(\cdot0003)^5 = \bar{4}{\cdot}477121 \times 5 = \overline{18}{\cdot}385605.$

(4) $2^x = 5$, and $\log_2 5 = x$, both express the same relation. $x \log 2 = \log 5 \;\therefore\; x = \dfrac{\log 5}{\log 2} = \dfrac{\cdot69897}{\cdot30103} = 2{\cdot}32.$

11. (1) and (2) Ex. 7, p. 198.

(3) Ex. 2, Appendix, Note B.

12. Art. 299.

(2) $\sqrt{1+m^2}\,(x^2+y^2) - 2cx - 2mcy = 0.$

To find the radius (Art. 302),

$$\left(x-\frac{c}{1+m^2}\right)^2 + \left(y-\frac{mc}{1+m^2}\right)^2 = \frac{c^2+m^2c^2}{1+m^2} = c^2.$$

Hence the radius is c.

13. Arts 308 and 311, and Ex. 1, p. 214.

14. (1) Art. 233 (5) substituting θ for A.

(2) $$\cos 3\theta = 4(\cos\theta)^3 - 3\cos\theta.$$

Now if θ denotes an angle of 18°, 3θ will contain 54°.

$$\therefore \sin 2\theta = \cos 3\theta,$$

and (231) $$\sin 2\theta = 2\sin\theta\cos\theta.$$

$$\therefore 2\sin\theta\cos\theta = 4(\cos\theta)^3 - 3\cos\theta.$$

$$2 \sin\theta = 4 (\cos\theta)^2 - 3 = 4 (1 - (\sin\theta)^2) - 3$$
$$= 1 - 4 (\sin\theta)^2. \quad \therefore 4 (\sin\theta)^2 + 2 \sin\theta = 1.$$

$$\therefore (\sin\theta)^2 + \frac{1}{2} \sin\theta = \frac{1}{4}.$$

$$\text{or } (\sin 18°)^2 + \frac{1}{2} \sin 18° = \frac{1}{4};$$

which, on solving, gives

$$\sin 18° = \frac{\sqrt{5}-1}{4}.$$

$$\text{Again (232) } \cos 36° = 1 - 2 (\sin 18°)^2 = 1 - 2\left(\frac{\sqrt{5}-1}{4}\right)^2$$
$$= \frac{1+\sqrt{5}}{4}.$$

15. To find in terms of the sides the $\sin\frac{A}{2}$, $\cos\frac{A}{2}$, and $\tan\frac{A}{2}$, see Art. 238.

For computing the angles by means of logarithms, the formula $\sin\frac{A}{2}$ must be used if the angle be near 90°, because the logarithms of the sines of arcs near 90° differ but little. If all the angles are required, the formulæ $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, and $\tan\frac{C}{2}$ are to be preferred, because the same factors are made use of in each case.

1874. *July 22nd.*

1. Arts. 41, 42.

2. $27\frac{3}{11}$ minutes past 5 o'clock. For explanation, see Ex. 15, p. 48.

3. (1) $\dfrac{2\sqrt{a^2-b^2}}{b}$. (2) $a^2 - b^2 + c^2 - 2ac$.

(3)
$$\frac{4ab + 2b^2\sqrt{-1} + 2\sqrt{4a^2b^2 + 2ab(2b^2\sqrt{-1}) + a^4 - b^4}}{a^2 + 2ab + b^2}.$$

4. If a, b, c, d, e, f, are placed in order of magnitude, a the greatest, f, the least, and the gradations are regular, the three quantities $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, are called ratios of greater inequality; $\frac{a}{b}$ has the

largest terms, and by taking the same quantity from each it is increased, and can be made equal to $\frac{c}{d}$.

Again $\frac{c}{d}$ can similarly be increased and made equal to $\frac{e}{f}$.

$$\therefore \frac{e}{f} > \frac{c}{d} \text{ and } \frac{c}{d} > \frac{a}{b}.$$

But $\frac{e}{f}$ is diminished by adding *equals* to each term, *à fortiori* $\frac{e}{f}$ is diminished by adding $\frac{c}{d}+\frac{a}{b}$ to each term,

$$\therefore \frac{e}{f} > \frac{a+c+e}{b+d+f} \text{ and } > \frac{a}{b}.$$

5. $1\frac{1}{2}$.

(2) Seven terms.

The sum of the first 6 terms is $1\frac{121}{243}$, or $\frac{1}{486}$ less than the sum.

" " 7 " $1\frac{364}{729}$, or $\frac{1}{1458}$ " "

6. Multiplying out, $a^2-b^2-c^2+d^2+2\,a\,d-2\,b\,c$

$$= a^2-b^2-c^2+d^2-2\,a\,d+2\,b\,c$$

$$\therefore 4\,a\,d = 4\,b\,c.$$

$$\text{or } a\,d = b\,c$$

$$\therefore (103)\quad a : b :: c : d.$$

7. Permutations. 1st, when $i < n$ and 2nd $i = n$.

1st. $P_i = n\,.\,\overline{n-1}\,.\,\overline{n-2}\,.\,.\,.\,.\,.\,\overline{n-i+2}\,.\,\overline{n-i+1}.$

2nd. $P_i = n\,.\,\overline{n-1}\,.\,\overline{n-2}\,.\,.\,.\,.\,.\,3\,.\,2\,.\,1.$

Combinations.

$$C_i = \frac{P_i}{i\,.\,\overline{i-1}\,.\,\overline{i-2}\,.\,.\,.\,.\,.\,3\,.\,2\,.\,1}.$$

(2) $\frac{12\,.\,11\,.\,10\,.\,9\,.\,8\,.\,7\,.\,6\,.\,5\,.\,4\,.\,3\,.\,2\,.\,1}{3\,.\,2\,.\,1\quad 4\,.\,3\,.\,2\,.\,1\,.\,5\,.\,4\,.\,3\,.\,2\,.\,1} = 27720.$

8. Present value $= \frac{100\,p}{c}$. (See Ex. 2, p. 101.)

(2) The value of the above annuity being 25 years' purchase $c = 4$, or money on these terms is worth 4 per cent. Then, by the question, the value of this annuity (169) will be,

$$\text{p.v.} = \frac{1}{1+r} + \frac{2}{(1+r)^2} + \frac{3}{(1+r)^3} + \ldots\ldots \textit{ ad infinitum.}$$

$$\text{Let } \left(\frac{1}{1+r}\right) = x.$$

$$= x + 2x^2 + 3x^3 + 4x^4 + \ldots\ldots\ldots \textit{ ad inf.}$$

$$= x(1 + 2x + 3x^2 + 4x^4 + \ldots\ldots \textit{ ad inf.}).$$

$$(\text{but } 1 + 2x + 3x^2 + \&\text{c.} = (1 + x + x^2 + x^3 + \&\text{c.}) \div (1-x),$$

$$\text{and } 1 + x + x^2 + x^3 + \&\text{c.} = \frac{1}{1-x}$$

$$\therefore 1 + 2x + 3x^2 + 4x^3 + \&\text{c.} = \frac{1}{(1-x)^2})$$

$$\therefore \text{p.v.} = \frac{x}{(1-x)^2} = \frac{\frac{1}{1+r}}{\left(1-\frac{1}{1+r}\right)^2} = \frac{\frac{1}{1+r}}{\left(\frac{r}{1+r}\right)^2}$$

$$= \frac{1+r}{r^2} \text{ or, in this case } \frac{1 \cdot 04}{\cdot 0016} = £650.$$

9. (α) $x = 5$ or 3. (β) $x = 16$; $y = 8$.

10. Let x = no. of years, p = population.

$$\frac{1}{30} - \frac{1}{40} = \frac{1}{120}, \text{ annual increase of population.}$$

$$\text{From the question, } p\left(1 + \frac{1}{120}\right)^x = 1\tfrac{1}{2}p.$$

$$\therefore \left(\frac{121}{120}\right)^x = \frac{3}{2}$$

$$x \log\left(\frac{121}{120}\right) = \log\frac{3}{2} = \cdot 1760913.$$

$$\text{Now } \log 120 = \log 2^2 + \log 3 + \log 10.$$

$$\log 121 = \log 11^2.$$

$$\text{and } \log\frac{121}{120} = \log 121 - \log 120 = 2 \cdot 082785 - 2 \cdot 079181 = \cdot 003604.$$

$$\therefore x = \frac{\cdot 1760913}{\cdot 0036041} = 48 \cdot 859.$$

The population will therefore be increased by *more than one half* in 49 years. *Ans.*

11. Ex. 5, p. 206.

12. *Proof by construction.* See Art. 349 and Fig. XXXVII., the centre is the origin (347, IV).

Make $a = 4 = C\,A$,

$b = 8 = C\,B$; Art. 343, $e^2 - 1 = \frac{b^2}{a^2} = \frac{64}{16} = 4. \therefore e = \sqrt{5} - 2$ nearly.

$$C\,S = ae \times 4\sqrt{5} = 9 \text{ nearly.}$$

Intercepts of the straight line $R\;T$ (R and T are the points in which it cuts the asymptotes), obtained from the given equation, are $x = 3\frac{1}{5}$, $y = \ -10\frac{2}{3}$.

To find the points A and P.

$$\left.\begin{array}{l} S\,A : A\,L \\ S\,P : P\,M \end{array}\right\} :: \left\{\begin{array}{l} 2 : 1 \\ 5 : 2\frac{1}{2}. \end{array}\right.$$

The co-ordinates of the point of contact P are $-x = 5$, $y = b$, and the co-ordinates of the points where $R\;T$ cuts the asymptotes—

$$\text{For } T,\ x = 2,\ y = -4.$$
$$R,\ x = 8,\ y = 16.$$

13. See Appendix, Note A, Case 3.

14. Suppose $\bar{a}$ and $\bar{b}$ given, then (236, 237).

$$2\,S - \overline{a+b} = c, \text{ and } 2\,S - c = a + b,$$

$$\text{also } S - c = \frac{(\text{area})^2}{S\,.\,\overline{S-a}\cdot\overline{S-b}};$$

$$\therefore S = a + b - \frac{(\text{area})^2}{S\,.\,\overline{S-a}\,.\,\overline{S-b}},$$

$$\text{but } c = 2\,S - \overline{a+b}$$

$$\therefore c = a + b - \frac{2\,(\text{area})^2}{S\,.\,\overline{S-a}\,.\,\overline{S-b}}.$$

The three sides are thus known, and the three angles can be found from them. See *First Case*, p. 151.

1875. *July 21st.*

1. $$\left.\begin{array}{l} 6930 = 2 \times 3^2 \times 5 \times 11 \times 7 \\ 1470 = 2 \times 3 \times 5 \times 7^2 \\ 5775 = 3 \times 5^2 \times 11 \times 7 \end{array}\right\} \text{See Art. 39.}$$

(1) $$\frac{1}{2\,.\,3^3\,.\,5\,.\,11\,.\,7} + \frac{1}{2\,.\,3\,.\,5\,.\,7\,.\,7} + \frac{1}{3\,.\,5^2\,.\,11\,.\,7}$$
$$= \frac{(7\,.\,5) + (3\,.\,11\,.\,5) + (7\,.\,3\,.\,2)}{2\,.\,3^3\,.\,11\,.\,7^2\,.\,5^2}.$$

(2) $$\sqrt{6930 \times 1470 \times 5775} = \sqrt{3^4\,.\,5^4\,.\,7^4\,.\,2^2\,.\,11^2}$$
$$= 3^2\,.\,5^2\,.\,7^2\,.\,2\,.\,11.$$

2. (1) 3·1622; and *four* more places by simple division.

(2) $\sqrt{\cdot 004} = \sqrt{\cdot 0004 \times 10} = \cdot 02\sqrt{10}$
$= \cdot 02 \times 3\cdot 1622 = \cdot 063245.$

3. $$\left(\frac{y-z}{x}+\frac{z-x}{y}+\frac{x-y}{z}\right)\left(\frac{x}{y-z}+\frac{y}{z-x}+\frac{z}{x-y}\right)$$
$$+\left(\frac{z-x}{x-y}-\frac{x-y}{z-x}\right)\left(\frac{y}{z}-\frac{z}{y}\right)+\left(\frac{x-y}{y-z}-\frac{y-z}{x-y}\right)\left(\frac{z}{x}-\frac{x}{z}\right)$$
$$+\left(\frac{y-z}{z-x}-\frac{z-x}{y-z}\right)\left(\frac{x}{y}-\frac{y}{x}\right).$$

(The following fractions are numbered for reference.)

(I) $$\left(\frac{y-z}{x}+\frac{z-x}{y}+\frac{x-y}{z}\right)\left(\frac{x}{y-z}+\frac{y}{z-x}+\frac{z}{x-y}\right)$$
$$= 3 + \overset{(1)}{\frac{x\,z-x^2}{y^2-y\,z}} + \overset{(2)}{\frac{x^2-y\,x}{y\,z-z^2}} + \overset{(3)}{\frac{y^2-y\,z}{x\,z-x^2}} + \overset{(4)}{\frac{x\,y-y^2}{z^2-x\,z}}$$
$$+ \overset{(5)}{\frac{y\,z-z^2}{x^2-x\,y}} + \overset{(6)}{\frac{z^2-x\,z}{x\,y-y^2}}.$$

(II) $$\left(\frac{z-x}{x-y}-\frac{x-y}{z-x}\right)\left(\frac{y}{z}-\frac{z}{y}\right) = \overset{(a)}{\frac{y\,z-x\,y}{x\,z-y\,z}} - \overset{(4)}{\frac{x\,y-y^2}{z^2-x\,z}}$$
$$- \overset{(6)}{\frac{z^2-x\,z}{x\,y-y^2}} + \overset{(b)}{\frac{x\,z-y\,z}{y\,z-x\,y}}.$$

(III) $$\left(\frac{x-y}{y-z}-\frac{y-z}{x-y}\right)\left(\frac{z}{x}-\frac{x}{z}\right) = \overset{(c)}{\frac{x\,z-y\,z}{x\,y-x\,z}} - \overset{(5)}{\frac{y\,z-z^2}{x^2-x\,y}}$$
$$- \overset{(2)}{\frac{x^2-x\,y}{y\,z-z^2}} + \overset{(a)}{\frac{x\,y-x\,z}{x\,z-y\,z}}.$$

(IV) $$\left(\frac{y-z}{z-x}-\frac{z-x}{y-z}\right)\left(\frac{x}{y}-\frac{y}{x}\right) = \overset{(b)}{\frac{x\,y-x\,z}{y\,z-x\,y}} - \overset{(1)}{\frac{x\,z-x^2}{y^2-y\,z}}$$
$$- \overset{(3)}{\frac{y^2-y\,z}{x\,z-x^2}} + \overset{(c)}{\frac{y\,z-x\,y}{x\,y-x\,z}} = 3 + \overset{(a)+(a)}{\frac{y\,z-x\,y+x\,y-x\,z}{x\,z-y\,z}}$$
$$+ \overset{(b)+(b)}{\frac{x\,z-y\,z+x\,y-x\,z}{y\,z-x\,y}} + \overset{(c)+(c)}{\frac{x\,z-y\,z+y\,z-x\,y}{x\,y-x\,z}}.$$

$$\left.\begin{array}{r}(1)+(1)=0\\(2)+(2)=0\\ \&c., \&c.\end{array}\right\}\text{ and }\left\{\begin{array}{l}(a)+(a)=-1\\(b)+(b)=-1\\(c)+(c)=-1\end{array}\right.$$

$$\therefore\ (I)+(II)+(III)+(IV)=3-1-1-1=0.$$

4. $\because$ the fractions are equal, the fraction formed by adding the numerators for a new numerator and the denominators for a new denominator is equal to each. The new fraction is

$$\frac{(a+b+c+d)(x+y+z)}{2\,(a+b+c+d)};$$

which $=\frac{x+y+z}{2}$, whether $a+b+c+d$ vanishes or not.

If $a+b+c+d=0$ the four fractions become

$$\frac{b\,x+c\,y+d\,z}{-2\,a}=\frac{c\,x+d\,y+a\,z}{-2\,b}=\frac{d\,x+a\,y+b\,z}{-2\,c}$$

$$=\frac{a\,x+b\,y+c\,z}{-2\,d},$$

where $b+c+d=-a,\ c+d+a=-b,\ d+a+b=-c,$
and $a+b+c=-d.$

Whence x must $=y$, and x and y must each $=z$. Therefore the value of each fraction is $\frac{3x}{2}$, or $\frac{3y}{2}$, or $\frac{3z}{2}$, or half the sum of their common value.

5. $6^5=7776$. It is here supposed that 1, 6, and 6, 1, are different throws.

6. Art. 114, III.

$$2\,S=d\,n^2+(2\,a-d)n.$$

$$\therefore\ d\,n^2+(2\,a-d)\,n-2\,S=0,\text{ and}$$

$$n=\frac{d-2\,a\pm\sqrt{(2\,a-d)^2+8\,d\,s}}{2\,d}.$$

(2) $n=13$ or 6, and both values satisfy the question. The last term is 12 or -9.

7. See p. 101, Question 3. (1) £47 18*s*. $6\frac{1}{4}d$.
(2) £48 1*s*. $6\frac{1}{2}d$.

8.
$$\left.\begin{array}{rl}\frac{x}{b+c}+\frac{y}{c-a} &=a+b\\ \frac{y}{c+a}+\frac{z}{a-b} &=b+c\\ \frac{x}{b-c}+\frac{z}{a+b} &=c+a\end{array}\right|\ \begin{array}{l}\text{put } a+b=m\\ b+c=n\\ c+a=p\\ a-b=q\\ b-c=r\\ c-a=s\end{array}$$

$$\frac{x}{n}+\frac{y}{s}=m \qquad s\,x+n\,y \qquad =m\,n\,s \;\ldots\ldots (1).$$

$$\frac{y}{p}+\frac{z}{q}=n \qquad q\,y+p\,z=n\,p\,q \;\ldots\ldots (2).$$

$$\frac{x}{r}+\frac{z}{m}=p \qquad m\,x \quad +r\,z=m\,p\,r \;\ldots\ldots (3).$$

$$(3) \times p \qquad p\,m\,x+p\,r\,z=m\,p^2\,r$$

$$(2) \times r \qquad \underline{q\,r\,y+p\,r\,z=n\,p\,q\,r}$$

$$(-) \qquad p\,m\,x-q\,r\,y=m\,p^2\,r-n\,p\,q\,r \;\ldots\ldots (4)$$

$$(1) \times q\,r \qquad q\,r\,s\,x+n\,q\,r\,y=m\,n\,q\,r\,s$$

$$(4) \times n \qquad m\,n\,p\,x-n\,q\,r\,y=m\,n\,p^2\,r-n^2\,p\,q\,r$$

$$(+) \qquad (m\,n\,p+q\,r\,s)\,x=n\,r\,(m\,q\,s+m\,p^2-n\,p\,q)$$

$$x=\frac{n\,r\,(m\,q\,s+m\,p^2-n\,p\,q)}{m\,n\,p+q\,r\,s};$$

Similarly

$$y=\frac{p\,s\,(n\,q\,r+m^2\,n-m\,p\,r)}{m\,n\,p+q\,r\,s}$$

$$z=\frac{m\,q\,(m\,r\,s+n^2\,p-m\,n\,s}{m\,n\,p+q\,r\,s}$$

restoring the original co-efficients,

$$x=\frac{\overline{b+c}\,.\,\overline{b-c}\{\overline{a+b}\,.\,\overline{a-b}\,.\,\overline{c-a}+\overline{a+b}\,.\,\overline{c+a}^2-\overline{b+c}\,.\,\overline{c+a}\,.\,\overline{a-b}\}}{\overline{a+b}\,.\,\overline{b+c}\,.\,\overline{c+a}+\overline{a-b}\,.\,\overline{b-c}\,.\,\overline{c-a}}$$

$$y=\frac{\overline{c+a}\,.\,\overline{c-a}\{\overline{b+c}\,.\,\overline{b-c}\,.\,\overline{a-b}+\overline{b+c}\,.\,\overline{a+b}^2-\overline{c+a}\,.\,\overline{a+b}\,.\,\overline{b-c}\}}{\overline{a+b}\,.\,\overline{b+c}\,.\,\overline{c+a}+\overline{a-b}\,.\,\overline{b-c}\,.\,\overline{c-a}}.$$

$$z=\frac{\overline{a+b}\,.\,\overline{a-b}\{\overline{c+a}\,.\,\overline{c-a}\,.\,\overline{b-c}+\overline{c+a}\,.\,\overline{b+c}^2-\overline{a+b}\,.\,\overline{b+c}\,.\,\overline{c-a}\}}{\overline{a+b}\,.\,\overline{b+c}\,.\,\overline{c+a}+\overline{a-b}\,.\,\overline{b-c}\,.\,\overline{c-a}}.$$

$$(2) \quad \frac{1}{x+\cfrac{1}{y-\cfrac{1}{x}}}=\frac{1}{x-\cfrac{1}{y-\cfrac{1}{x}}} \text{ or } \frac{x\,y-1}{x^2\,y}=\frac{x\,y-1}{x^2\,y-2\,x}.$$

which cannot be equal unless $x=\frac{1}{y}$.

$$\text{If } x=\frac{1}{y},$$

$$\frac{1}{y}\left(1-\frac{1}{x}\right)=x-1=1.$$

whence $x=2$; $y=\frac{1}{2}$.

9. Reducing, $x+\frac{b}{a}=\sqrt{\left(\frac{b}{a}\right)^2-\frac{c}{a}}$; and if $\frac{c}{a}=\left(\frac{b}{a}\right)^2$, the values of x will be equal.

(2) $x=-\frac{b}{a}$. (3) $x^2-2\,a\,x+a^2-\beta^2=0$.

See Articles 176—82.

10.
$$\left.\begin{array}{l}\left(\frac{24}{x}\right)^2+(y-4)^2=65\\ \left(\frac{12}{x}\right)^2+9=(5y-20)^2\end{array}\right\}.$$

$$\therefore\ (y-4)^2-36=65-4\,(5\,y-20)^2.$$
$$=65-100\,(y+4)^2.$$
$$101\,(y-4)^2=101.$$
$$y-4=\pm 1.$$
$$y=5 \text{ or } 3.$$
$$\left(\frac{24}{x}\right)^2=64\quad\therefore\ x=\pm 3.$$

11. The logarithm to a given base of any number is the index of the power to which the base must be raised to obtain the given number. Thus in the equation $64=2^6$, 6 is the logarithm of 64 to the base 2 (189).

(2) Any series of logarithms to the same base, constitutes a *system*, e.g. Napier's and Briggs' systems. See (192).

(3) See answer to Qu. 9 (5), 1862, p. 46 of this Key.

12.
$$\begin{array}{r}\bar{1}\cdot 5793262\\ 3\\ \hline \bar{2}\cdot 7379786\\ \bar{2}\cdot 9495976\\ \hline 5)\bar{1}\cdot 7883810\\ \hline \bar{1}\cdot 9576762\end{array} = \log \cdot 9071439.\quad \textit{Ans.}$$
$$\begin{array}{r}43\\ \hline 48)190(39\\ 460\\ 28\end{array}$$

13. Ex. 6, p. 208.

14. Ex. 3, p. 215.

15. Ex. 2, pages 235—7.

16. Art. 206. $\theta = 57{\cdot}296^\circ = 3437'\tfrac{3}{4} = 206264''\tfrac{3}{4} = 12{,}375{,}888'''$.
$= 57^\circ\ 17'\ 44''\ 48'''$.

17. Art. 233, p. 142 (4) and (5).

18. Art. 241. $a = c\,\dfrac{\sin A}{\sin(\pi - \overline{A+B})}$; $b = c\,\dfrac{\sin B}{\sin(\pi - \overline{A+B})}$.

(2) $h = b \sin A = c\,\dfrac{\sin A \sin B}{\sin(\pi - \overline{A+B})}$.

(3) Area $= \dfrac{1}{2}\, b\, c \sin A = \dfrac{1}{2} c^2 \dfrac{\sin A \sin B}{\sin(\pi - \overline{A+B})}$.

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42